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Coordinatization of Triangulated Jordan Systems

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We obtain partial coordinatizations of any quadratic Jordan triple system containing a triangle (e_1, u, e_2), i.e., a "split" $J_0 = ke_1 \oplus ku \oplus ke_2 \cong H_2(k)$, such that $e = e_1 + e_2$ is invertible and u is faithful (in the sense that $L(x_1, e_1)u = 0 \to x_1 = 0$; this is automatic if J is nondegenerate or $\frac{1}{2} \in k$). We identify a hermitian matrix subsystem $J_h = J_2(e_1) \oplus Du \oplus J_2(e_2) \cong H_2(D, D_0, \pi, {}^-)$ and a Clifford subsystem $J_q = K_1 \oplus N \oplus K_2 \cong J(q, S, C_0)$. We show that a simple J must be all hermitian or all Clifford (coincides with J_h or J_q), yielding a shorter proof Osborn's Capacity 2 Theorem for Jordan algebras. We also can coordinatize most unital bimodules for hermitian matrix systems.

The impetus for this study came from two sources. In presenting a self-contained treatment of Jordan structure theory, N. Jacobson [4] sought a shorter proof of the Osborn Capacity 2 Theorem; the proof is not short for linear Jordan algebras [2] and is quite involved for quadratic Jordan algebras [3]. In coordinatizing Jordan triple systems, the second author needed a version of the Capacity 2 Theorem making no semisimplicity restriction on the ambient system and without assuming that the tripotents e_i are division tripotents [10]. By avoiding semisimplicity assumptions, such a formulation also coordinatizes unital bimodules.

In the triangulated case one cannot expect as precise a coordinatization as for rank $n \geq 3$: In the latter case if J is a Jordan algebra then $J \cong$

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$H_n(D, D_0)$ and all bimodules have a uniform structure [2, 7], whereas in the former a $J = ke_1 \oplus ku \oplus ke_2$ can have bimodules of hermitian matrix type $H_2(D, D_0, \pi, ^-)$ or Clifford type $J(q, S, C_0)$. Thus we expect a triangulated system to be a mixture of hermitian matrix and Clifford parts. We will isolate these parts and obtain a separate coordinatization for each one.

1. Generalities about Triangulated Jordan Triple Systems

Throughout we assume that J is a Jordan triple system over a ring k, with Peirce decomposition

$$J = J_{11} \oplus J_{12} \oplus J_{22} \tag{1.1}$$

relative to orthogonal tripotents (e_1, e_2). (For all unexplained notions the reader is referred to [4, 5, 7, or 9].) We sometimes write

$$J_{11} = J_1, \qquad J_{12} = M, \qquad J_{22} = J_2.$$

Always the index $i \in \{1, 2\}$, in which case $j \in \{1, 2\}$ is given by $j = 3 - i$, so $(i, j) = (1, 2)$ or $(2, 1)$. For $x_i \in J_i$, $m \in M$, $x \in J$ we set

$$\begin{aligned} Q_j(m) &= P(m)e_i, \qquad x_i \cdot m = L(x_i)m = \{x_i e_i m\}, \\ \bar{x} &= P(e)x, \qquad \text{where} \quad e = e_1 + e_2. \end{aligned} \tag{1.2}$$

A general product $P(x)y$ in J is given by

$$P(x_1 + m + x_2)(y_1 + n + y_2) = z_1 + r + z_2,$$

where

$$\begin{aligned} z_i &= P(x_i)\, y_i + P(m)\, y_j + \{x_i n m\} \\ r &= P(m)n + \{x_1 y_1 m\} + \{x_2 y_2 m\} + \{x_1 n x_2\}. \end{aligned}$$

Using well-known Peirce multiplication rules and standard Jordan identities, most of these products can be written in terms of Q_i's, L's, and $^-$. More precisely, we have

$$P(x_i)\, y_i \in J_i, \quad P(m)\, y_i \in J_j \qquad \text{(not reducible)} \tag{1.3.1}$$

$$P(m)n = Q_i(m, \bar{n}) \cdot m - Q_j(m) \cdot \bar{n} \tag{1.3.2}$$

$$\{mnx_i\} = Q_i(m, x_i \cdot \bar{n}) = \{m, \bar{x}_i \cdot n, e_i\} \tag{1.3.3}$$

$$\{mx_i n\} = Q_j(m, \bar{x}_i \cdot n) = Q_j(n, \bar{x}_i \cdot m) \tag{1.3.4}$$

(1.3.5) $$\{x_i y_i m\} = x_i \cdot (\bar{y}_i \cdot m)$$

(1.3.6) $$\{x_i m y_j\} = x_i \cdot (y_j \cdot \bar{m}) = y_j \cdot (x_i \cdot \bar{m})$$

(1.3.7) $$e_i \cdot m = m, \qquad P(x_i)\, y_i \cdot m = x_i \cdot (\bar{y}_i \cdot (x_i \cdot m)),$$
$$x_i^2 \cdot m = x_i \cdot (x_i \cdot m) \qquad \text{for} \quad x_i^2 = P(x_i)e_i$$

(1.3.8) $$P(x_i \cdot m)\, y_i = P(m)\, \overline{P(x_i)\, y_i}, \quad Q_j(x_i \cdot m) = P(m)\, \overline{P(x_i)e_i}$$
$$P(x_i \cdot m)\, y_j = P(x_i)\, \overline{P(m)\, y_j}, \quad Q_i(x_i \cdot m) = P(x_i)\, \overline{Q_i(m)}$$

(1.3.9) $$P(\{x_i y_i m\}, m) z_j = L(x_i, y_i)\, P(m) z_j,$$
$$Q_i(\{x_i y_i m\}, m) = L(x_i, y_i)\, Q_i(m)$$

(1.3.10) $$Q_i(m) \cdot m = Q_j(m) \cdot m$$

(1.3.11) $\bar{\ }$ is an automorphism of period 2 on J stabilizing the J_{ij} (reducing to $P(e_i)$ on J_i, $P(e_1, e_2)$ on M) with $\bar{e}_i = e_i$

(1.3.12) $$\overline{Q_i(m)} = Q_i(\bar{m}).$$

Notice that a linear subspace $K = K_{11} \oplus K_{12} \oplus K_{22}$ with $K_{ij} \subset J_{ij}$ is a subsystem if

(1.3′) $$K_{ii} = \bar{K}_{ii}, \qquad K_{12} = \bar{K}_{12},$$
$$P(K_{ii})K_{ii} \subset K_{ii}, \quad P(K_{12})K_{ii} \subset K_{jj}, \quad K_{ii} \cdot K_{12} \subset K_{12}$$

(in case $e_i \in K_i$ we also have "iff").

1.4. PROPOSITION. *If $J = J_{11} \oplus J_{12} \oplus J_{22}$ relative to e_1, e_2 and $J' = J'_{11} \oplus J'_{12} \oplus J'_{22}$ relative to e'_1, e'_2, then φ: $J \to J'$ satisfying $\varphi(e_i) = e'_i$, $\varphi(J_{ij}) = J'_{ij}$ is a homomorphism iff it acts as a homomorphism on the products*

$$P(x_i)\, y_i,\ P(m)\, y_i, \text{ and } x_i \cdot m.$$

If φ commutes with involutory automorphisms $*$, $*'$ on J, J' with $e_i^* = e_j$, $e_i'^{*'} = e'_j$, it suffices if φ preserves $P(x_1)\, y_1$, $P(m)\, y_1$, $x_1 \cdot m$. ∎

The involutions $*$ we are interested in come from triangles. We say J is *triangulated* if it has a *triangle* (e_1, e_2, u) (u is then called *triangular*) satisfying

(1.5) $u \in J_{12}$ has $P(e)u = u$, $P(u)e = e$ (so $P(u)u = u = \bar{u}$, $Q_i(u) = e_i$, $T_i(u) = 2e_i$, u is a tripotent with $J_2(u) = J$, and $x \to x^{\mu} = P(u)x$ is an involutory automorphism of J), where in addition to the quadratic forms Q_i of (1.2) we introduce linear forms

$$T_i(m) = Q_i(u, m) = \{ue_j m\}. \tag{1.2'}$$

Important examples of triangulated systems are matrix algebras $H_2(D, D_0)$ ($e_1 = \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$, $e_2 = \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)$, $u = \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$) (see Section 2), or split Clifford algebras $J(Q, e) = ke_1 \oplus ku \oplus ke_2$ for $Q(u) = 1$, $Q(e_1, u) = Q(e_2, u) = Q(e_1) = Q(e_2) = 0$, $Q(e_1, e_2) = 1$ (see Section 3). We remark that there is another, analytic example: any JBW-algebra without direct summand of type I is triangulated. This is exactly the assertion of Schafer's halving lemma [1, 5.2.14].

1.6. TRIANGULATED IDENTITIES. *If $J = J_1 \oplus M \oplus J_2$ is triangulated by (e_1, u, e_2) then we have the properties*

(1.6.1) $\quad P(e)\,P(u) = P(u)\,P(e) = *$ *is an automorphism of J of period 2, $u^* = u$, $e_i^* = e_j$ (so $J_i^* = J_j$)*

$$\begin{aligned} T_i(m)^* &= T_i(m^*) = T_j(m), \\ \overline{T_i(m)} &= T_i(\bar{m}), \qquad Q_i(m)^* = Q_j(m^*) \end{aligned} \tag{1.6.2}$$

$$m^* = T_i(m) \cdot u - m \tag{1.6.3}$$

$$x_i^* \cdot u = (x_i \cdot u)^* = x_i \cdot u \tag{1.6.4}$$

$$T_i(x_i \cdot u) = 2x_i, \qquad T_i(\{x_i y_i u\}) = \{x_i y_i e_i\}. \tag{1.6.5}$$

If C denotes the subalgebra of $\mathrm{End}_k(M)$ *generated by $C_0 = L(J_1)$, then C carries a reversal involution*

$$(L(x_1) \cdots L(x_n))^\pi = L(x_n) \cdots L(x_1) \qquad (x_i \in J_1) \tag{1.6.6}$$

given explicitly by

$$c^\pi = L(u, \overline{cu}) - c^*, \quad c^* = P(u)\,cP(u), \tag{1.6.7}$$

such that C_0 is an ample subspace:

$$c + c^\pi = L(T_1(cu)) \tag{1.6.8}$$

$$cL(y_1)c^\pi = L(P(cu)\,y_1^\mu). \tag{1.6.9}$$

We have the relations

$$(cu)^* = c^*u = c^\pi u \tag{1.6.10}$$

$$Q_1(cu, m) = T_1(c^*m) = T_1(cm^*), \quad Q_2(cu, m) = T_2(c^\pi m) \tag{1.6.11}$$

$$(1.6.12)\qquad \begin{aligned}(c^* - c)m &= (T_1(m)c^\pi - T_1(cm))u,\\ (x_i^* - x_i)\cdot m &= (T_i(m)x_i - T_i(x_i\cdot m))u\end{aligned}$$

$$(1.6.13)\qquad C[C, C]\,CM \subset Cu.$$

For $\Gamma(x_1; m) = L(T_1(x_1\cdot m)) - L(T_1(m))\,L(x_1) \in C$ *we have*

$$(1.6.14)\qquad \begin{aligned}\Gamma(x_1; m)\,\Gamma(x_1; m)^\pi\, m = {}& L(Q_2(m))[L(x_1), \Gamma(x_1; m)]u\\ &+ L(x_1)[L(Q_1(m), L(x_1)]m\\ &+ [L(x_1), L(P(m)x_1^\mu)]m \in Cu.\end{aligned}$$

Proof. (1) $P(u) = P(P(e)u) = P(e)\,P(u)\,P(e)$ and $P(e)^2 = I$ give $P(e)\,P(u) = P(u)\,P(e)$, so their common value $*$ is involutory with $e_i^* = e_j$, $u^* = u$. Also $P(u)^2 = P(u)\,P(e)\,P(u)\,P(e) = * \circ * = I$ shows $J = J_2(u)$, so in particular $L(u, u) = 2I$.

(2) $Q_i(m)^* = (P(m)e_j)^* = P(m^*)e_j^* = P(m^*)e_i = Q_j(m^*)$, $T_i(m)^* = \underline{Q_i(m, u)}^* = Q_j(m^*, u^*) = Q_j(m^*, u) = T_j(m^*)$, also $T_i(\bar{m}) = Q_i(\bar{m}, \bar{u}) = \overline{Q_i(m, u)} = \overline{T_i(m)}$. We have $T_j(m^*) = L(u, e_i)\,P(u)\bar{m} = P(P(u)e_i, u)\bar{m} = P(e_j, u)\bar{m} = Q_j(\bar{\bar{m}}, u) = T_j(m)$. (Note that we do not have $Q_i(m^*) = Q_i(m)$ in general, e.g., in $M_2(D)$, $Q_1((\begin{smallmatrix}0 & b\\ c & 0\end{smallmatrix})^*) - Q_1(\begin{smallmatrix}0 & b\\ c & 0\end{smallmatrix}) = (\begin{smallmatrix}[c,b] & 0\\ 0 & 0\end{smallmatrix}) \neq 0$ if D is not commutative.)

(3) $m^* = P(u)\bar{m} = Q_i(u, m)\cdot u - Q_j(u)\cdot m = T_i(m)\cdot u - e_j\cdot m = T_i(m)\cdot u - m$.

(4) $(x_i\cdot u)^* = x_i^*\cdot u^* = x_i^*\cdot u = L(u, e_j)\,P(u)\bar{x}_i = P(P(u)e_j, u)\bar{x}_i = P(e_i, u)\bar{x}_i = x_i\cdot u$.

(5) $Q_i(\{x_i y_i u\}, u) = L(x_i, y_i)\,Q_i(u)$ (by (1.3.9)) $= \{x_i y_i e_i\}$. (Note that this shows again $T_j(m^*) = T_j(T_j(m)\cdot u - m) = 2T_j(m) - T_j(m) = T_j(m)$.)

Before we show that the reversal involution π is well-defined on C, note that the relations (10)–(13) hold for all monomials $c = L(x_1)\cdots L(x_n)$, since for (10), $(cu)^* = c^*u^* = L(x_1^*)\cdots L(x_n^*)\,u^* = L(x_1^*)\cdots L(x_{n-1}^*)\,L(x_n)u$ (by (4)) $= L(x_n)\,L(x_1^*)\cdots L(x_{n-1}^*)u$ (by (1.3.6)) $= \cdots = L(x_n)\,L(x_{n-1})\cdots L(x_1)u = c^\pi u$. Then for (11), $Q_1(cu, m) = Q_1(c^{*\pi}u, m) = Q_1(u, c^*m)$ (peeling off one factor $L(x_i^*)$ at a time via (1.3.4)) $= T_1(c^*m) = T_1((cm^*)^*) = T_1(cm^*)$, and similarly $Q_2(cu, m) = Q_2(u, c^\pi m) = T_2(c^\pi m)$. For (12), $c^*m - cm = c^*(m + m^*) - ((cm)^* + cm) = c^*T_1(m)u - T_1(cm)u$ (by (3)) $= T_1(m)\,c^*u - T_1(cm)u$ (by (1.3.6)) $= (T_1(m)c^\pi - T_1(cm))u$ (by (10)). For (13), since $C[C, C]C$ is generated as ideal by all $[L(x_1), L(y_1)]$, it suffices to show $[x_1, y_1]M \subset Cu$; but $[x_1, y_1]M = [x_1 - x_1^*, y_1]M$ (x_1^*, y_1 commute by (1.3.6)) $\subset Cu - y_1\cdot Cu$ (by (12)) $\subset Cu$.

The fact that π is well-defined on C comes from its explicit expression

(7); to see that the right side does indeed produce the reversal involution on $c=L(x_1)\cdots L(x_n)$ we induct on n, the case $n=0$ being trivial ($L(u,u)=2I$ by (1.5)), and if true for $d=L(x_2)\cdots L(x_n)$ of length $n-1$ the result holds for $c=L(x_1)\,d$ because

$$(c^\pi+c^*)m-\{u,\overline{cu},m\}$$

$$=(d^\pi L(x_1)+L(x_1^*)\,d^*)m-\{u,\overline{x_1\cdot du},m\}$$

$$=((d^\pi+d^*)\,L(x_1)+(L(x_1^*)-L(x_1))\,d^*)m$$
$$-\{u,\overline{x_1\cdot du},m\}\quad\text{(by (1.3.6))}$$

$$=\{u,\overline{du},x_1\cdot m\}-\{u,\bar{x}_1\cdot\overline{du},m\}+(T_1(d^*m)\cdot(x_1\cdot u)$$
$$-T_1(x_1\,d^*m)\cdot u)\quad\text{(by induction for } d \text{ and (12))}$$

$$=-Q_1(du,m)\cdot(x_1\cdot u)+Q_1(du,x_1\cdot m)u$$
$$+(Q_1(d^*m,u)\cdot x_1\cdot u-Q_1(d^*x_1m,u)\cdot u)$$

(from $(L(\{x_1e_1m\},\bar{n})-L(m,\{e_1x_1\bar{n}\}))u=(-L(\{x_1\bar{n}m\},e_1)+L(x_1,\{e_1m\bar{n}\}))u=-Q_1(x_1\cdot n,m)\cdot u+x_1(Q_1(n,m)\cdot u)=Q_1(n,x_1\cdot m)\cdot u-Q_1(n,m)\cdot(x_1\cdot u)$ by linearized (1.3.9))

$$=(-Q_1(du,m)+Q_1(m,d^{*\pi}u))\cdot(x_1\cdot u)$$

$$+(Q_1(x_1m,du)-Q_1(x_1m,d^{*\pi}u))u$$

$$=0\qquad\text{(from (10) } c^{*\pi}u=cu).$$

Thus π is well-defined.

For (8), $cm+c^\pi m=cm+\{u,\overline{cu},m\}-c^*m$ (by (7)) $=cm+(Q_1(u,cu)m+Q_1(m,cu)u-Q_2(u,m)cu)-c^*m$ (by (1.3.2)) $=cm+T_1(cu)m+T_1(cm^*)u-T_2(m)\,cu-(cm^*)^*$ (by (11)) $=T_1(cu)m+c(m-T_2(m)u)+cm^*$ (by (3), (1.3.6)) $=T_1(cu)\cdot m$ (by (3)).

It suffices to prove (9) and its linearization $c\to c,d$ for monomials; if $c=L(x_1)\cdots L(x_r)$ is a monomial then $L(x)\,L(y)\,L(x)=L(U(x)\,y)$ (for $U(x)=P(x)\,P(e)$ in the Jordan algebra $J^{(e)}$ using (1.3.7)) shows $cL(y)c^\pi=L(U(x_1)\cdots U(x_r)y)=L(U(x_1)\cdots U(x_r)\,U(u)\,U(u)y)=L(U(L(x_1)\cdots L(x_r)u)\,U(u)\,y)=L(P(cu)\,P(e)\,P(u)\,P(e)\,y)=L(P(cu)\,P(u)\,y)$ (by $\bar{u}=u$). For the mixed term $cL(y)\,d^\pi+dL(y)c^\pi=L(T_1(cL(y)\,d^\pi u))$ (by (8)), where $P(cu,du)\,y^\mu=Q_1(y^*cu,du)=Q_1(cy^*u,du)=Q_1(cyu,du)$ (by (4)) $=T_1(cy(du)^*)$ (by (11)) $=T_1(cyd^*u)=T_1(cyd^\pi u)$ (by (10)).

Finally, we show (14). Since $\Gamma(x_1;m)\,\Gamma(x_1;m)^\pi = L(Q_1(\Gamma(x_1;m)\cdot u))$ by (9) we first compute

$$
\begin{aligned}
&Q_1(\Gamma(x_1;m)\cdot u)\\
&\quad = Q_1(x_1\cdot m - x_1^*\cdot m) \quad \text{(by (12))}\\
&\quad = Q_1(x_1\cdot m) - Q_1(x_1\cdot m, x_1^*\cdot m) + Q_1(x_1^*\cdot m)\\
&\quad = P(x_1)\,Q_1(\bar{m}) - \{m, x_1^\mu, x_1\cdot m\} + P(m)\,P(x_1^\mu)e_2\\
&\qquad\quad \text{(by (1.3.4) and (1.3.8))}\\
&\quad = P(x_1)\,Q_1(\bar{m}) - \{x_1 e_1 P(m) x_1^\mu\} + P(m)(P(x_1)e_1)^\mu
\end{aligned}
$$

by (1.3.9), (1.5). Using the general formula

$$(P(m)\,y_1^\mu)\cdot m = \{Q_2(m), y_1^\mu, m\} = Q_2(m)\cdot(y_1^*\cdot m)$$

we obtain

$$
\begin{aligned}
&\Gamma(x_1;m)\,\Gamma(x_1;m)^\pi\, m\\
&\quad = Q_1(\Gamma(x_1;m)u)\cdot m \qquad \text{(by the above)}\\
&\quad = x_1\cdot(Q_1(m)\cdot(x_1\cdot m) - P(m)x_1^\mu)\cdot m)\\
&\qquad - (P(m)x_1^\mu)\cdot(x_1\cdot m) + Q_2(m)\cdot((P(x_1)e_1)^*\cdot m)\\
&\quad = x_1\cdot(Q_1(m)\cdot(x_1\cdot m) - Q_2(m)\cdot(x_1^*\cdot m))\\
&\qquad - (P(m)x_1^\mu)\cdot(x_1\cdot m) + Q_2(m)\cdot(x_1^*\cdot(x_1^*\cdot m))\\
&\quad = x_1\cdot(Q_2(m)\cdot(x_1\cdot m - x_1^*\cdot m)) + Q_1(m)\cdot(x_1\cdot m)\\
&\qquad - x_1\cdot(Q_2(m)\cdot m)) + Q_2(m)\cdot(x_1^*\cdot(x_1^*\cdot m) - x_1\cdot(x_1^*\cdot m))\\
&\qquad + Q_2(m)\cdot(x_1\cdot(x_1^*\cdot m)) - (P(m)x_1^\mu)\cdot(x_1\cdot m)\\
&\quad = x_1\cdot(Q_2(m)\cdot(\Gamma(x_1;m)u))\\
&\qquad + L(x_1)[L(Q_1(m)), L(x_1)]m\\
&\qquad + Q_2(m)\cdot(x_1^*\cdot(x_1^*\cdot m - x_1\cdot m)) + x_1\cdot(Q_2(m)\cdot(x_1^*\cdot m))\\
&\qquad - (P(m)x_1^\mu)\cdot(x_1\cdot m) \qquad \text{(by (1.3.10))}\\
&\quad = Q_2(m)(L(x_1)\,\Gamma(x_1;m) - L(x_1^*)\,\Gamma(x_1;m))u\\
&\qquad + L(x_1)[L(Q_1(m)), L(x_1)]m\\
&\qquad + [L(x_1), L(P(m)x_1^\mu)]m \qquad \text{(by the general formula),}
\end{aligned}
$$

which equals the right side of (14) since $L(x_1^*)\,\Gamma(x_1;m)u = \Gamma(x_1;m)\,L(x_1^*)u = \Gamma(x_1;m)\,L(x_1)u$ by (10). Finally, $\Gamma(x_1;m)\,\Gamma(x_1;m)^\pi\, m \in Cu$

follows from (13) and $Q_2(m) \cdot du = d \cdot (Q_2(m) \cdot u) = d \cdot (Q_1(m^*)^* \cdot u) = d \cdot (Q_1(m) \cdot u) \in Cu$ for $d \in C$. ∎

1.7. *Remark.* A more philosophical proof derives the fundamental identity (1.6.7) as the coefficient of t in the relation

$$(7') \qquad U(n)(cn^{-1}) = c^{\pi} n, \qquad U(n) = P(n)\, P(e)$$

for $n = u - tm$, $n^{-1} = u + tU(u)m + t^2 U(u)\, U(m)u + \cdots$ in $J[[t]]$. Relation (7′) is just a generalization of the special case $n = u$ (1.6.10) and holds for all invertible $n \in J_{12}$: it is proved by induction on the length of the monomial c, $c = 1$ being trivial $(U(m)m^{-1} = m)$, and for the induction step $U(n)(cL(x_1)n^{-1} = U(n)(cL(U(n^{-1})x_1)n) = U(n)\, L(U(n^{-1})x_1)\, cn = L(x_1)\, U(n) Q_2(n^{-1}) cn = L(x_1) U(n) c Q_2(n)^{-1} n = L(x_1)\, U(n) cn^{-1} = L(x_1) c^{\pi} n$. Alternately, we can simply "isotope" the special case $n = u$: in $\tilde{J} = J^{(v)}$ for $v = e_1 + Q_2(n)^{-1}$ we have a triangle $\tilde{e}_1 = e_1$, $\tilde{e}_2 = Q_2(n)$, $\tilde{u} = n$ with $\tilde{J}_i = J_i$, $\tilde{L}(x_i) = L(x_i)$, $\tilde{*} = \tilde{U}(\tilde{u}) = U(n)\, U(v) = U(n)$ on J_1, $= U(n)^{-1}$ on J_2, $= U(n)\, Q_2(n)^{-1}$ on M, so $c^{\pi} n = (L(x_1) \cdots L(x_r))^{\pi}\, n = (\tilde{L}(x_1) \cdots \tilde{L}(x_r))^{\pi}\, \tilde{u} = \tilde{U}(\tilde{u})(\tilde{L}(x_1) \cdots \tilde{L}(x_r) \tilde{u}) = U(n)\, L(Q_2(n)^{-1})\, L(x_1) \cdots L(x_r) n = U(n)\, L(x_1) \cdots L(x_r)\, Q_2(n)^{-1}\, n = U(n)\, cn^{-1}$.

Replacing c by $cL(Q_1(n))$ yields the polynomial form

$$(7'') \qquad U(n)\, cn = Q_1(n)\, c^{\pi} n.$$

We have established this for all invertible n, so by general philosophical principles (see the Laurent Trick (4.2)) it must be true for all n. Notice that in previous versions the result was only used for invertible n, which required intricate verifications that there were enough such elements. ∎

In any Peirce-2-space of a Jordan triple we can define squares of commutators: For $x, y, z \in J_2(c)$, $c = c^3$, put

$$[x, y]^2 := \{x, c, P(y)\bar{x}\} - P(x)\, P(\bar{y})c - P(y)\, P(\bar{x})c \qquad (P(c)x = \bar{x}),$$

or more generally

$$P([x, y]) := L(x, \bar{y})\, L(y, \bar{x}) - L(P(x)\, P(\bar{y})c, c) - P(x)\, P(\bar{y}) - P(y)\, P(\bar{x}),$$

also

$$[[x, y], z] := \{x\bar{y}z\} - \{y\bar{x}z\}.$$

These definitions are motivated by the following fact: For any unital specialization σ of the Jordan algebra $J_2(c)^{(c)}$ in End V (σ: $J_2(c) \to$ End V linear with $\sigma(P(x)\bar{y}) = \sigma(x)\sigma(y)\sigma(x)$, $\sigma(c) = I$) we have

$$\sigma([x, y]^2) = [\sigma(x), \sigma(y)]^2,$$

$$\sigma(P([x, y])z) = [\sigma(x), \sigma(y)]\,\sigma(z)[\sigma(x), \sigma(y)] \tag{1.8}$$

$$\sigma([[x, y], z]]) = [[\sigma(x), \sigma(y)], \sigma(z)].$$

Applying this to $J_1 = J_2(e)$ in a triangulated triple and $\sigma = L$ yields

(1.9) If the Jordan ideal of J_1 generated by $P([J_1, J_1])J_1$ is all of J_1 (e.g., if J_1 is simple and $[J_1, J_1]^2 \neq 0$), then $M = Cu$.

For, by assumption, e_1 is the sum of elements which are built up from $P([J_1, J_1])J_1$ using multiplication operators. By (1.8) the image of this sum under the specialization $\sigma = L$ has the form $L(e_1) = \sum c_{i1}[L(x_{1i}), L(y_{1i})]\,c_{i2}$, therefore $M = C[C_0, C_0]\,CM \subset Cu$ by (1.6.13). ▮

We have a general formula

$$Q_i([L(x_i), L(y_i)]m) + P(m)[\bar{x}_i, \bar{y}_i]^2 = 0 \tag{1.10}$$

relating commutator-squares: $Q_i(x_i \cdot (y_i \cdot m)) + Q_i(y_i \cdot (x_i \cdot m)) - Q_i(x_i \cdot (y_i \cdot m), y_i \cdot (x_i \cdot m)) = P(y_i \cdot m)\,P(\bar{x}_i)e_i + P(x_i \cdot m)\,P(\bar{y}_i)e_i - Q_i(y_i \cdot m, (P(x_i)\bar{y}_i) \cdot m)$ (by (1.3.8) and (1.3.4)) $= P(m)(P(\bar{y}_i)\,P(x_i) + P(\bar{x}_i)\,P(y_i) - P(\bar{y}_i, P(\bar{x}_i)\,y_i))e_i$. ▮

1.11. *Remark.* If C is commutative, then, by (1.8), Ker L contains all $P([x_1, y_1])z_1$ and $[[x_1, y_1], z_1]$ (and in characteristic 2 all $\{x_i y_i z_i\}$). If L is faithful ($L(x_1) = 0 \to x_1 = 0$) these elements must vanish in J_1, so if in addition J_1 is nondegenerate, we know from [8] that $J_1 \subset \Omega_1^+$ for a commutative associative Ω_1 without nilpotents. ▮

Quite often (see below) faithfulness of L can be checked on the triangular element u alone: We call *u faithful* if it is faithful to J_1:

$$z_1 \cdot u = 0 \to z_1 = 0 \quad \text{in} \quad J_1 \qquad (u\text{-faithfulness}).$$

1.12. FAITHFULNESS CRITERION. *A triangular element u is automatically faithful if any of the following hold*:

(1) *J is *-special: $P(x)y = x\bar{y}x$ and $P(e)y = \bar{y}$ in A*;

(2) *J_1 is nondegenerate (e.g., if J is nondegenerate)*;

(3) $\frac{1}{2} \in k$;

(4) *$M = Cu$ and J has zero extreme radical;*

(5) *J_1 contains no trivial "central" *-ideal $Z_1 = \bar{Z}_1 \lhd J_1$:*

$$P(Z_1)J_1 = \{Z_1 J_1 J_1\} = 2Z_1 = 0.$$

Proof. (1) If $\{z_1 e_1 u\} = 0$ then $0 = e_1\{z_1 e_1 u\}u = (e_1 z_1 e_1)u^2$ (since $e_1 u e_1 = 0$) $= e_1 z_1 e_1 e_1 = e_1 z_1 e_1 = \bar{z}_1$ in $e_1 A e_1$. (2) $0 = P(z_1 \cdot u)J_2 = P(z_1)P(u)J_2$ (by (1.3.8)) $= P(z_1)J_1 = P(z_1)J$, so such z_1 is trivial. (3) $0 = T(z_1 \cdot u) = 2z_1$ (by (1.6.5)). (4) $z_1 \cdot u = 0 \to z_1 \cdot M = z_1 \cdot Cu = z_1 C^* u$ (by (1.6.10)) $= C^* z_1 u = 0$ (note here that extreme radical $= \{z_1 + z_2;\ z_i \cdot M = 0\}$). (5) $Z_1 = \{z \in J_1;\ z_1 \cdot u = 0\}$ is such an ideal: $\bar{z}_1 \cdot u = \overline{z_1 \cdot u} = 0$, $P(Z_1)J = 0$ (above (2)), $2Z_1 = 0$ (above (3)), $P(J_1)Z_1 \subset Z_1$ ($P(x_1)z_1 \cdot u = x_1 \cdot (\bar{z}_1 \cdot (x_1 \cdot u)) = x_1 \cdot (x_1^* \cdot (\bar{z}_1 \cdot u)) = 0$), $\{z_1 J_1 J_1\} = P(z_1, J_1)P(e_1)P(u)J_2 = P(z_1 \cdot u, J_1 \cdot u)J_2$ (by (1.3.8)) $= 0$. ∎

1.13. Example of Unfaithful u. Let $C = k1 \oplus kz$ be the ring of dual numbers in characteristic 2 (so C has unit 1 and $z^2 = 0$), and make $J = Ce_1 \oplus (ku \oplus Cv) \oplus Ce_2$ into a C-module by letting C operate on e_1, v, and e_2 by left multiplication and requiring $\mathrm{Ann}_C(u) = kz$. Putting $q(c_1 e_1 \oplus (\gamma u \oplus cv) \oplus c_2 e_2) = c_1 c_2 + \gamma^2 + c^2 w + \gamma c z$ ($w \in C$ arbitrary) defines a quadratic form q: $J \to C$, and S: $J \to J$: $c_1 e_1 \oplus (\gamma u \oplus cv) + c_2 e_2 \to c_2 e_1 \oplus (\gamma u \oplus cv) \oplus c_1 e_2$ is an involutory isometry. Then $P(x)y = q(x, Sy)x - q(x)Sy$ defines on J a Jordan triple system (actually a Jordan algebra with unit $e = e_1 + e_2$) with a triangle (e_1, u, e_2) and an unfaithful u: $z \cdot u = 0$, yet $z \neq 0$. (This is a special example of a Clifford system to be discussed in Section 3.)

In some instances we might even have *C-faithfulness*:

$$cu = 0 \text{ for } c \in C \to c = 0 \text{ in End } M \qquad (C\text{-faithfulness}).$$

1.14. C-Faithfulness Criterion. *A triangular element u is C-faithful if any of the following hold:*

(1) *J is *-special $P(x)y = x\bar{y}x$ and $P(e)y = \bar{y}$ for an associative A with involution $^-$;*

(2) *$M = Cu$;*

(3) *J_1 contains no *-ideal $Z_1 = \bar{Z}_1 \ntrianglelefteq J_1$ with $P([J_1, J_1])J_1 + [[J_1, J_1], J_1] \subset Z_1$ (or in characteristic 2 with $\{J_1 J_1 J_1\} \subset Z_1$).*

Proof. (1) If $c = \sum L(x_1) \cdots L(x_n)$ has $cu = 0$ then $0 = e_1(cu)u = e_1(\sum x_1 \cdots x_n u)u$ (since $L(x_1)m = x_1 e_1 m + m e_1 x_1 = x_1 m + m x_1$ and any term $A_{11} u A_{11}$ vanishes) $= e_1(\sum x_1 \cdots x_n)u^2 = \sum x_1 \cdots x_n$, so also $0 = (\sum x_1 \cdots x_n)m = e_1(\sum L(x_1) \cdots L(x_n)m)$. Dually $0 = u(cue_1)$ implies $0 =$

$\sum x_n \cdots x_1$ and $0 = m(\sum x_n \cdots x_1) = (\sum L(x_1) \cdots L(x_n)m)e_1$, so $0 = e_1(cm) + (cm)e_1 = L(e_1)(cm) = cm$, i.e., $c = 0$. (2) As in (1.12), $cu = 0 \to cM = 0$, so $c = 0$ in End M. For (3) let $Z = \{c \in C; cu = 0\}$, $Z_1 = \{z_1 \in J_1;\ ZL(z_1) = 0\}$. Then $\bar{Z} = Z$ implies $\bar{Z}_1 = Z_1$ ($ZL(\bar{z}_1) = \bar{Z}L(z_1) \subset \overline{ZL(z_1)} = 0$); as we saw in (2), Z satisfies $ZCu = 0$, hence

$$(*) \qquad ZC[L(x_1), L(y_1)] = 0$$

since $[L(x_1), L(y_1)]M \subset Cu$ by (1.6.13). Therefore in characteristic 2, $ZL(\{x_1 y_1 z_1\}) = Z(L(x_1)\, L(\bar{y}_1)\, L(z_1) + L(z_1)\, L(\bar{y}_1)\, L(x_1)) = 2ZL(x_1)\, L(\bar{y}_1)\, L(z_1) = 0$ and $\{x_1 y_1 z_1\} \in Z_1$, in general $P([x_1, y_1])z_1$ and $[[x_1, y_1], z_1] \in Z_1$ by (1.8) and $(*)$. Z_1 is an ideal since it is a linear space which is inner ($ZL(P(z_1)x_1) = ZL(z_1)\, L(\bar{x}_1)\, L(z_1) = 0$) and outer (by commutativity $(*)$ $ZL(P(x_1)z_1) = ZL(x_1)\, L(\bar{z}_1)\, L(x_1) = ZL(\bar{z}_1)\, L(x_1)^2 = 0$, $ZL(\{x_1 y_1 z_1\}) = Z\{L(x_1) L(\bar{y}_1) L(z_1)\} = ZL(z_1)(L(\bar{y}_1) L(x_1) + L(x_1) L(\bar{y}_1)) = 0$). If J_1 contains no such $Z_1 \lhd J_1$ then we must have $Z_1 = J_1$, $e_1 \in Z_1$, $Z = ZL(e_1) = 0$, and u is C-faithful. ∎

1.15. PROPOSITION. *Let J be triangulated. Then J is nondegenerate iff J_1 and Q_1 are nondegenerate, and J is simple iff J_1 is simple and Q_1 is nondegenerate.*

Proof. Suppose J_1 and Q_1 are nondegenerate; then via $(*)$ J_2 and Q_2 are also nondegenerate. If $z = z_1 + m + z_2$ is trivial then so are the $z_i = P(e_i)^2 z$, hence $z_i = 0$ by nondegeneracy of J_i, and $z = m$; then $0 = P(z)e_j = P(m)e_j = Q_i(m)$, and $Q_i(m, M) = 0$ since each $Q_i(m, n)$ is trivial in J_i ($P(Q_i(m, n)) = P(\{me_j n\}) = P(m)\, P(e_j)\, P(n) + P(n)\, P(e_j)\, P(m) + L(n, e_j)\, P(m)\, L(e_j, n) - P(P(n)\, P(e_j)m, m) = 0$ from $P(m) = P(z) = 0$ and $P(e_j)m = 0$), so $m \in \operatorname{Rad} Q_i = 0$ shows $m = 0$, $z = 0$, and J is nondegenerate.

Conversely, suppose J is nondegenerate. Then it is well known that J_1 inherits nondegeneracy ($P(z_1)J = P(z_1)J_1$). To see $\operatorname{Rad} Q_1 = 0$ consider $m \in \operatorname{Rad} Q_1$; then $m^* \in \operatorname{Rad} Q_2$ by (1.6.2), and $m^* = -m$ since $T_1(m) = Q_1(m, u) = 0$, so $m \in \operatorname{Rad} Q_1 \cap \operatorname{Rad} Q_2$. Then $P(m)M = 0$, and we will show $m = 0$ by showing $P(m)J = P(m)J_1 + P(m)J_2$ vanishes. But

$$\begin{aligned}
P(P(m)x_i)J_j &= P(m)\, P(x_i)\, P(m)\, P(u)J_j^{\mu} \\
&= P(m)[P(\{x_i mu\}) + P(P(x_i)\, P(m)u, u) \\
&\quad - L(x_i, m)\, P(u)\, L(m, x_i) - P(u)\, P(m)\, P(x_i)]\, J_j^{\mu} \\
&= -P(m)\, P(u)\, P(m)\, P(x_i)J_j^{\mu} \\
&= -P(P(m)u)\, P(x_i)J_j^{\mu} = 0
\end{aligned}$$

from $P(m)u = 0$ and $L(x_i, m) = 0$ on M ($L(x_i, m)M = Q_i(x_i \cdot \bar{m}, M) =$

$\overline{Q_i(\bar{x}_i \cdot m, \bar{M})} = \overline{\bar{x}_i \cdot Q_i(m, \bar{M})} - \overline{Q_i(m, \bar{x}_i \cdot \bar{M})} = 0$ for $m \in \operatorname{Rad} Q_i$). Thus $P(m)x_i$ is trivial in J_j, and therefore vanishes.

(Alternately, $z_j = P(m)x_i$ will vanish if it is trivial in J_j, hence if $P(P(z_j)J_j) = 0$. But

$$\begin{aligned}
P(P(z_j)J_j) &= P(z_j)\, P(J_j)\, P(z_j) \\
&= P(z_j)\, P(J_j)\, P(m)\, P(x_i)\, P(m) \\
&= [P(\{z_j J_j m\}) + P(P(z_j)\, P(J_j)m, m) - P(m)\, P(J_j)\, P(z_j) \\
&\quad - L(z_j, J_j)\, P(m)\, L(J_j, z_j)]\, P(x_i)\, P(m) \\
&= P(z_j \cdot (\bar{J}_j \cdot m))\, P(x_i)\, P(m),
\end{aligned}$$

so it suffices if $z_j \cdot M = 0$, which follows from $\{z_j e_j M\} = \{P(m)x_i, e_j, M\} = P(L(M, e_j)m, m)x_i - P(m)\, L(e_j, M)x_i \in \{Q_i(m, M), x_i, m\} - P(m)M = 0$.)

If J is simple then J_1 is simple by [6] (and J is automatically nondegenerate). Conversely, if J_1 is simple so is J_2, and they are automatically nondegenerate, and so $\operatorname{Rad} Q_1 = 0$ guarantees J is nondegenerate. If $I \lhd J$ is a nonzero ideal then $I = I_1 \oplus N \oplus I_2$, so by simplicity of J_1 either $I_1 = 0$ (then $I_2 = I_1^* = 0$ and $Q_1(N) + Q_1(N, J_{12}) \subset I \cap J_1 = 0 \to N \subset \operatorname{Rad} Q_1 = 0$, so $I = 0$) or $I_1 = J_1$ (then $I_2 = J_2$, $J_{12} = \{e_1 e_1 J_{12}\} \subset N$, so $I = J$). This shows J is simple. ∎

Remark. A more careful analysis works for any $J = J_1 \oplus J_{12} \oplus J_2$: (i) J is nondegenerate if J_i are nondegenerate and the Q_i are nondegenerate and (ii) J is simple if J_i are simple, Q_i are nondegenerate, and $J_{12} \neq 0$. ∎

We will encounter polarized Jordan triple systems $T = T^+ \oplus T^-$ satisfying

$$P(T^\sigma)T^\sigma = 0 = L(T^\sigma, T^\sigma)T, \qquad P(T^\sigma)T^{-\sigma} \subset T^\sigma$$

for $\sigma = \pm$. In this case $\mathscr{V} = (T^+, T^-)$ is a Jordan pair. If T is also a triangulated triple, then $e = e^+ + e^-$ is invertible, so $(T^+, T^-) \cong (J, J)$ for $J = T^{+(e^-)}$ via $(\mathrm{Id}, P(e^-))$: $(J, J) \to (T^+, T^-)$ and the Jordan algebra J is triangulated by (e_1^+, u^+, e_2^+).

1.16. Polarization Criterion. *For a faithfully triangulated J the following are equivalent*:

(i) *J is polarized*;

(ii) *J_1 is polarized (as subsystem of J) or, equivalently, J_2 is polarized*;

(iii) *C_0 is a polarized Jordan triple system with respect to*

$$P(L(x_1))\, L(y_1) = L(P(x_1)\, y_1) = L(x_1)\, L(\bar{y}_1)\, L(x_1).$$

In this case the Jordan pair (J^+, J^-) *is isomorphic to* $(\tilde{J}, \tilde{J})$, *where* $\tilde{J}$ *is a faithfully triangulated Jordan algebra. Moreover,* (J^+, J^-) *is simple (or non-degenerate) iff the same holds for* $\tilde{J}$.

Proof. Every polarization $J = J^+ \oplus J^-$ induces one of J_i, since $e_i = e_i^+ + e_i^-$, where (e_i^+, e_i^-) is an idempotent of the Jordan pair $\mathscr{V} = (J^+, J^-)$, and we have $J_i = J_2(e_i) = J_2^+ \oplus J_2^-$ for $\mathscr{V}_2(e_i^+, e_i^-) = (J_2^+, J_2^-)$, so (i) $\Rightarrow$ (ii). By faithfulness and (1.3.7), $J_1 \cong C_0$ under $x_1 \Rightarrow L(x_1)$, showing (ii) $\Leftrightarrow$ (iii), and therefore it only remains to prove (ii) $\Rightarrow$ (i), which is the main assertion of this criterion.

Asssume J_1 is polarized. Then the polarization is given by $I_1 = I_1^+ \oplus I_1^-$ for $I_1^\varepsilon = P(e_1^\varepsilon)\, P(e_1^{-\varepsilon})$ $(e_1 = e_1^+ \oplus e_1^- \in J_1^+ \oplus J_1^-)$. The automorphism $*$ carries this over to a polarization of J_2 given by $I_2^\varepsilon = P(e_2^\varepsilon)\, P(e_2^{-\varepsilon})$ for

$$(1) \qquad e_2^\varepsilon = (e_1^\varepsilon)^* = (e_1^{-\varepsilon})^\mu.$$

We get a linear decomposition

$$J_{12} = J_{12}^+ \oplus J_{12}^-, \qquad J_{12}^\varepsilon = I_{12}^\varepsilon(J_{12}) \qquad \text{for} \quad I_{12}^\varepsilon = L(e_1^\varepsilon, e_1^{-\varepsilon})$$

since the I_{12}^ε are idempotents $(L(e_1^\varepsilon, e_1^{-\varepsilon})^2 = L(P(e_1^\varepsilon)e_1^{-\varepsilon}, e_1^{-\varepsilon}) + 2P(e_1^\varepsilon)\, P(e_1^{-\varepsilon}) = L(e_1^\varepsilon, e_1^{-\varepsilon}) + 0$ on $J_{12})$ which are supplementary, hence orthogonal, since $I = L(e_1, e_1) = L(e_1^+, e_1^-) + L(e_1^-, e_1^+)$ because $L(e_1^\varepsilon, e_1^\varepsilon) = 0$ from the special case $x = e_1^{-\varepsilon}$, $y = e_1^\varepsilon$ of

$$(3) \qquad \{x e_1^\varepsilon y\} = 0 = P(e_1^\varepsilon)\, y \Rightarrow L(P(e_1^\varepsilon)x, y) = 0 = L(y, P(e_1^\varepsilon)x)$$

$(L(P(e_1^\varepsilon)x, y) = -L(P(e_1^\varepsilon)\, y, x) + L(e_1^\varepsilon, \{x, e_1^\varepsilon, y\}) = 0)$. The case $x = e_1^{-\varepsilon}$, $y \in J^\varepsilon$ of (3) (note $\{e_1^{-\varepsilon}, e_1^\varepsilon, y\} = 0$ for any $y \in J_2$, any $y \in J_1^\varepsilon$ by polarization, and $y \in J_{12}^\varepsilon$ since $L(e_1^{-\varepsilon}, e_1^\varepsilon) = I^{-\varepsilon}$ on J_{12}) yields

$$L(e_1^\varepsilon, J^\varepsilon) = L(J^\varepsilon, e_1^\varepsilon) = 0.$$

This shows $\{J_1^\varepsilon, e_1^\varepsilon, J^\varepsilon\} = P(e_1^\varepsilon) J^\varepsilon = 0$, so by (3),

$$(4) \qquad L(J_1^\varepsilon, J^\varepsilon) = L(J^\varepsilon, J_1^\varepsilon) = 0.$$

Next, for $x^\varepsilon, y^\varepsilon \in J_{12}^\varepsilon$,

$$\begin{aligned} \{e_1^{-\varepsilon} x^\varepsilon y^\varepsilon\} &= \{e_1^{-\varepsilon}, x^\varepsilon, L(y^\varepsilon, e_1^{-\varepsilon}) e_1^\varepsilon\} \\ &= -\{L(y^\varepsilon, e_1^{-\varepsilon}) e_1^{-\varepsilon}, x^\varepsilon, e_1^\varepsilon\} + L(y^\varepsilon, e_1^{-\varepsilon})\{e_1^{-\varepsilon} x^\varepsilon e_1^\varepsilon\} \\ &\quad + \{e_1^{-\varepsilon}, L(e_1^{-\varepsilon}, y^\varepsilon) x^\varepsilon, e_1^\varepsilon\} = 0 \end{aligned}$$

(by (4) and the polarization of J_1), therefore $L(x^\varepsilon, y^\varepsilon) = L(\{x^\varepsilon e_1^{-\varepsilon} e_1^\varepsilon\}, y^\varepsilon) + L(e_1^\varepsilon, \{e_1^{-\varepsilon} x^\varepsilon y^\varepsilon\}) = [L(x^\varepsilon, e_1^{-\varepsilon}), L(e_1^\varepsilon, y^\varepsilon)] = 0$ by (4), so

$$(5) \qquad L(J_{12}^\varepsilon, J_{12}^\varepsilon) = 0.$$

Now we are able to establish

$$(6) \qquad L(e_2^\varepsilon, e_2^{-\varepsilon}) = L(e_1^\varepsilon, e_1^{-\varepsilon}) \qquad \text{on} \quad J_{12}$$

since $L(e_2^\varepsilon, e_2^{-\varepsilon}) x_{12} = P(u)\, L(e_1^{-\varepsilon}, e_1^\varepsilon)\, P(u) x = P(u)[P(\{e_1^{-\varepsilon} e_1^\varepsilon u\}, u) x - P(u)\, L(e_1^\varepsilon, e_1^{-\varepsilon}) x] = P(u)\{u\ {}^\varepsilon x u\} - P(u)^2 x^\varepsilon = P(u)\{u x^\varepsilon u\} - x^\varepsilon$ (by (5) twice) $= 2P(u)^2 x^\varepsilon - x^\varepsilon = x^\varepsilon = L(e_1^\varepsilon, e_1^{-\varepsilon}) x_{12}$. This shows that * respects the polarization of J_{12} as well, so (4) and (5) yield $L(J_i^\varepsilon, J^\varepsilon) = L(J^\varepsilon, J_i^\varepsilon) = L(J_{12}^\varepsilon, J_{12}^\varepsilon) = 0$, thus

$$(\mathrm{I}) \qquad \{J^\varepsilon J^\varepsilon J\} = 0.$$

The other polarization condition

$$(\mathrm{II}) \qquad P(J^\varepsilon) J^\varepsilon = 0, \qquad P(J^\varepsilon) J^{-\varepsilon} \subset J^\varepsilon$$

is easy to establish, because the polarization is given by structural transformations

$$(7) \qquad \begin{aligned} I^\varepsilon &= B(e_1^{-\varepsilon}, e_1^\varepsilon)\, B(e_2^{-\varepsilon}, e_2^\varepsilon) \\ &= B(e_2^{-\varepsilon}, e_2^\varepsilon)\, B(e_1^{-\varepsilon}, e_1^\varepsilon), \qquad I^{\varepsilon *} = I^{-\varepsilon} \end{aligned}$$

and thus we immediately have $P(J^\varepsilon) J^\eta = P(I^\varepsilon(J))\, I^\eta(J) = I^\varepsilon P(J)\, I^{-\varepsilon} I^\eta J$ vanishes if $\eta \neq -\varepsilon$, and lands in $I^\varepsilon(J) = J^\varepsilon$. Indeed, to see that (7) holds, simply observe that $B(e_i^{-\varepsilon}, e_i^\varepsilon)$ reduces on J_i to $I - L(e_i^{-\varepsilon}, e_i^\varepsilon) + P(e_i^{-\varepsilon})\, P(e_i^\varepsilon) = I - 2I^{-\varepsilon} + I^{-\varepsilon} = I^\varepsilon$ $[L(e_i^{-\varepsilon}, e_i^\varepsilon) = L(e_i^{-\varepsilon}, e_i^\varepsilon) I^{-\varepsilon} = L(e_i, e_i) I^{-\varepsilon} = 2I^{-\varepsilon}$ by polarization of J_i], on J_{12} to $I - L(e_i^{-\varepsilon}, e_i^\varepsilon) = I - I^{-\varepsilon}$ (by (6)) $= I^\varepsilon$, and on J_j to I. ∎

One of the main purposes in obtaining a coordinatization theorem for non-semisimple triple systems is to be able to describe bimodules. Recall that in general a *bimodule* for a Jordan triple system J over k is a k-module M together with compositions $p(x)m$ and $l(z, y)m$ linear in z, y, m and quadratic in x having the property that the split null extension

$$E = J \oplus M, \qquad P(x \oplus m)(y \oplus n) = P(x) y \oplus (p(x) n + l(x, y) m)$$

is again a Jordan triple system containing J as a subsystem and M as a trivial ideal ([5, Sect. 2] or [7]). If J contains an invertible tripotent e

(as is the case for triangulated triples) then M splits into three different sub-bimodules, namely the Peirce spaces relative to e:

$$E = E_2(e) \oplus E_1(e) \oplus E_0(e) \qquad \text{for} \quad J = J_2(e),$$

where

> $E_2(e) = J \oplus M_2(e)$ and $M_2(e) = \{m \in M; p(e)^2 m = m\}$ is a *unital bimodule*,
>
> $E_1(e) = M_1(e) = \{m \in M;\ l(e,e)m = m\}$ is a *special* bimodule, $p(x) = 0$ on $M_1(e)$ (see lemma below), and
>
> $E_0(e) = M_0(e) = \{m \in M; l(e,e)\, m = 0 = p(e)\, m\}$ is *trivial*, $l(x, y) = p(x) = 0$ on $M_0(e)$,

because $J \subset E_2(e)$.

1.17. Special Bimodule Criterion. *The special bimodules of J with $J = J_2(e)$ are nothing else but unital specializations of the Jordan algebra $J^{(e)}$ ($U_x = P_x P_e$) in endomorphism algebras*:

Any unital specialization σ of $J^{(e)}$ in End M *(σ: $J \to$* End M *linear satisfying $\sigma(U_x y) = \sigma(x)\,\sigma(y)\,\sigma(x)$ and $\sigma(e) = I$) makes M into a special bimodule via*

$$l(x, y) = \sigma(x)\,\sigma(\bar{y}), \qquad p(x) = 0 \qquad (\bar{y} = P_e y).$$

Conversely, if M is a special bimodule for J, then $\sigma(x) = l(x, e)$ is a unital specialization of $J^{(e)}$.

Proof. Since special bimodules always have $p(x) = 0$, they are characterized by the existence of a bilinear map l: $J \times J \to$ End M satisfying

$$(*) \qquad \begin{gathered} l(x, y)\, l(x, z) = l(P_x y, z), \quad l(z, x)\, l(y, x) = l(z, P_x y), \\ l(e, e) = I. \end{gathered}$$

It is easily verified that $l(x, y) = \sigma(x)\,\sigma(\bar{y})$ satisfies $(*)$. Conversely, given a special bimodule M of J we have $\sigma(x)\,\sigma(y)\,\sigma(x) = l(x, e)\, l(y, e)\, l(x, e) = l(x, P_e y)\, l(x, e)$ (by $(*)$) $= l(P_x P_e y, e)$ (by $(*)$) $= \sigma(U_x y)$. ∎

We will later describe unital bimodules for hermitian matrix systems.

2. Hermitian Coordinatization

In this section we characterize the triangulated hermitian matrix systems and derive a coordinatization theorem for them. An (associative) *coordinate system* $(D, D_0, \pi, {}^-)$ consists of a unital associative algebra D with *involution* π, and an *automorphism* of period 2 commuting with π, together with a stable π-*ample* subspace $D_0 \subset H(D, \pi)$ $[\bar{D}_0 \subset D_0, 1 \in D_0, d_0^\pi = d_0, dd_0 d^\pi \in D_0$ for all $d \in D, d_0 \in D_0]$. The *hermitian matrix system* $H = H_2(D, D_0, \pi, \ )$ consists of all 2×2 matrices over D which are hermitian $(X = X^{\pi t})$ and have diagonal entries in D_0, with triple product $P(X)Y = X\bar{Y}^{\pi t}X = X\bar{Y}X$. Thus H is spanned by elements

$$
(2.1) \qquad \begin{aligned} & d_0[ii] = d_0 E_{ii}, \\ & d[12] = dE_{12} + d^\pi E_{21} = d^\pi[21] \qquad (d \in D, d_0 \in D_0) \end{aligned}
$$

with products

$$
(2.2) \qquad \begin{aligned} P(d_0[ii])\, c_0[ii] &= d_0 \bar{c}_0\, d_0[ii] \\ P(d[ij])\, c_0[jj] &< d\bar{c}_0\, d^\pi[ii] \\ P(d[12])\, c[12] &= d\bar{c}^\pi\, d[12] \\ \{d_0[ii]\; c_0[ii]\; d[ij]\} &= d_0\bar{c}_0\, d[ij] \\ \{d_0[ii]\; d[ij]\; c_0[jj]\} &= d_0\, \bar{d}c_0[ij] \\ \{d[ij]\; b[ji]\; d_0[ii]\} &= (d\bar{b}\, d_0 + d_0\bar{b}^\pi\, d^\pi)[ii]. \end{aligned}
$$

Such a system is triangulated by $e_1 = 1[11]$, $e_2 = 1[22]$, $u = 1[12]$ with

$$
\overline{\begin{pmatrix} d_0 & d \\ d^\pi & c_0 \end{pmatrix}} = \begin{pmatrix} \bar{d}_0 & \bar{d} \\ \bar{d}^\pi & \bar{c}_0 \end{pmatrix}, \quad \begin{pmatrix} d_0 & d \\ d^\pi & c_0 \end{pmatrix}^\mu = \begin{pmatrix} \bar{c}_0 & \bar{d}^\pi \\ \bar{d} & \bar{d}_0 \end{pmatrix}, \quad \begin{pmatrix} d_0 & d \\ d^\pi & c_0 \end{pmatrix}^* = \begin{pmatrix} c_0 & d^\pi \\ d & d_0 \end{pmatrix}
$$

$$
\begin{aligned} & d_0[ii] \cdot d[ij] = d_0\, d[ij] \qquad (d_0[11] \cdot d[12] = d_0\, d[12], \\ & \qquad\qquad d_0[22] \cdot d[12] = d\, d_0[12]) \end{aligned}
$$

$$
(2.3) \qquad \begin{aligned} & Q_i(d[ij]) = d\, d^\pi[ii] \qquad (Q_1(d[12]) = d\, d^\pi[11], \\ & \qquad\qquad Q_2(d[12]) = d^\pi\, d[22]) \\ & T_i(d[ij]) = (d + d^\pi)[ii]. \end{aligned}
$$

We say H is *diagonal* if the diagonal coordinates D_0 generate all coordinates D; the archetypal example of a nondiagonal matrix system is when

$D =$ quaternations, $D_0 = k$. In this case $H_2(D, D_0, \pi)$ is really a Clifford system $ke_1 \oplus D \oplus ke_2$ relative to the quadratic form $q(d) = d\bar{d}$ in D; see Section 3.

2.4. HERMITIAN COORDINATIZATION THEOREM. *For any faithfully triangulated triple system J, the hermitian subsystem*

$$J_h = J_1 \oplus C(u) \oplus J_2$$

is a diagonal hermitian matrix system

$$J_h \cong H_2(D, D_0, \pi, {}^-) \qquad \text{under} \quad x_1 \oplus d(u) \oplus y_2 \xrightarrow{\varphi} \begin{pmatrix} L(x_1) & d \\ d^\pi & L(y_2^*) \end{pmatrix}$$

for

$$\begin{aligned} D &= C|_{Cu} \qquad (C \subset \mathrm{End}_k(M) \text{ the subalgebra generated by } L(J_1)), \\ D_0 &= C_0|_{Cu} \quad (C_0 = L(J_1) \subset C), \\ d^\pi &= d^{\mathit{reverse}} \\ \bar{d} &= P(e) \circ d \circ P(e). \end{aligned}$$

We have $J_h = J$ iff $M = C(u)$; this happens if C contains an invertible commutator (e.g., if some $[L(x_1), L(y_1)]$ is invertible on M) or, more generally, the Jordan ideal generated by all $P([x_1, y_1])z_1$ is all of J_1 (see (1.9)).

Proof. $(C, C_0, \pi, {}^-)$ is a coordinate system by (1.6), so its restriction $(D, D_0, \pi, {}^-)$ to $Cu \subset M$ is one too. By definition of D we have D-faithfulness

$$du = 0 \Rightarrow d = 0 \qquad \text{in} \quad D \qquad (\text{i.e., } dCu = 0)$$

since $dCu = dC^*u$ (by 1.6.10)) $= C^*\, du = 0$. Because $y_2 \to y_2^*$ is bijective and u is faithful to J_1, we conclude that φ is a well-defined bijection. It clearly commutes with ${}^-$ and also with $*$ (use $d^*u = d^\pi u$ by (1.6.10) for the latter). So, to see that φ is a homomorphism it is enough by 1.4 to show that φ preserves products $P(x_1)y_1$, $P(m)x_2$, $x_1 \cdot m$ $(m = du)$, i.e.,

$$\begin{aligned} L(P(x_1)y_1) &= L(x_1)\, L(\bar{y}_1)\, L(x_1), \\ L(P(du)x_1^\mu) &= dL(x_1)\, d^\pi, \\ \varphi(x_1 \cdot du) &= L(x_1)\, d. \end{aligned}$$

The first equation follows from (1.3.7), the second from (1.6.9), and the third from the definition.

Thus $J_h \cong H_2(D, D_0, \pi, {}^-)$. Clearly we have $J_h = J$ iff $Cu = M$. To see $Cu = M$ when C has invertible commutators we apply (1.6.13). ▮

2.5. *Remark.* Our coordinatization applies only to that part of M generated out of u via J_1. This need not hold for arbitrary $H_2(D, D_0, \pi, {}^-)$: if D_0 does not generate D then $Cu < M$, and we cannot recover the associative structure of D from the Jordan structure of M (eg., D = quaternions, $D_0 = k1$, π = standard involution, ${}^-$ = id: here H_2 looks like $ke_1 \oplus M \oplus ke_2 = J(q, S)$ for a quadratic form, and the fact that the quadratic form permits an associative composition is not a Jordan property). ▮

Our first application of the Hermitian Coordinatization Theorem is a description of unital bimodules of $H_2(D, D_0, \pi, {}^-)$.

2.6. HERMITIAN BIMODULE COORDINATIZATION. *If $J = H_2(D, D_0, \pi, {}^-)$ is a diagonal hermitian matrix system then any unital J-bimodule M which*

(i) *is faithful: $m_1 \cdot u = 0 \Rightarrow m_1 = 0$ (i.e., $l(u, e_1)\, p(e_1)$ is injective on M) and*

(ii) has $M_{12} = M_1 \cdot Cu$, *where $M = M_1 \oplus M_{12} \oplus M_2$ is the Peirce decomposition relative to the natural triangle in J (this just means $E_{12} = C_E \cdot u$ for the split null extension $E = J \oplus M$)*

has the form

$$M \cong H_2(N, N_0, \pi, {}^-),$$

where $(N, N_0, \pi, \ )$ is a unital coordinate bimodule for $(D, D_0, \pi, {}^-)$ (an associative bimodule N for D with commuting linear maps π, ${}^-$ of period 2 extending π, ${}^-$ on D: $(d \cdot n)^\pi = n^\pi \cdot d^\pi$, $(dn)^- = \bar{d}\bar{n}$, $(nd)^- = \bar{n}\bar{d}$, N_0 is a ${}^-$-invariant π-ample subspace of $H(N, \pi)$: $dN_0 d^\pi \subset N_0$ for all $d \in D$, and $n + n^\pi \in N_0$ for all $n \in N$).

Condition (ii) *is automatic if the Jordan ideal of D_0 generated by all $P([x_1, y_1])z_1$ $(x_1, y_1, z_1 \in J_1)$ is all of D_0, e.g., if $(D_0, {}^-)$ is simple and $[x_1, y_1]^2 \neq 0$.*

Proof. By (i), $u \in E_{12}$ is faithful for $C_0 = \{L(d_1 \oplus m_1)\} \subset \operatorname{End} E_{12}$ and (ii) allows us to apply the Hermitian Coordinatization Theorem: $E \cong H_2(C, C_0, \pi, {}^-)$ for a coordinate system $(C, C_0, \pi, {}^-)$. In particular, u is C-faithful. Therefore, C is actually the split null extension of D by $N = \{c \in C;\ cu \in M_{12}\} = l(M_1)\, D = Dl(M_1)$ using $l(m_1) \circ l(x_1) = l(x_1 \cdot m_1)$ to move $l(M_1)$ to left or right, and $l(M_1)^2 = 0$. The reversal involution π and the automorphism ${}^-$ on C respect this splitting: $D^\pi = D = \bar{D}$, $N^\pi = N = \bar{N}$, and $\pi \mid D$, ${}^- \mid D$ are the given involution and automorphism on D since D_0

generates D. The ample subspace splits correspondingly: $C_0 = D_0 \oplus N_0$, $N_0 = L(M_1)$. The ampleness condition is $(d+n)(d_0+n_0)(d^\pi + n^\pi) = dd_0\, d^\pi \oplus (dn_0\, d^\pi + [dd_0 n^\pi + n\, d_0\, d^\pi]) \in D_0 \oplus N_0$ and is equivalent to $dN_0\, d^\pi \subset N_0$ and $n^\pi + n \in N_0$ since then $dd_0 n^\pi + n\, d_0\, d^\pi = (n\, d_0\, d^\pi)^\pi + (n\, d_0\, d^\pi)$ automatically falls into N_0. ∎

If $\mathscr{S}$ is a set of mappings on an algebra A we call I an *ideal* of $(A, \mathscr{S})$ (written as $I \lhd (A, \mathscr{S})$) iff I is an ideal of A left invariant by all $s \in \mathscr{S}$:

$$I \lhd A \qquad \text{and} \qquad sI = I \qquad \text{for all } s \in \mathscr{S}.$$

We say A is *$\mathscr{S}$-simple* or *$(A, \mathscr{S})$ is simple* if $(A, \mathscr{S})$ is not trivial and has no proper ideals. For $\mathscr{S} = \varnothing$ this is the usual simplicity, for $\mathscr{S} = \{*\}$, $*$ an involution, we have the familiar $*$-simplicity. Ideals of $(D, \mathscr{S}) = (D, \pi, {}^-)$ appear if one studies ideals of hermitian matrix systems:

2.7. Proposition. *The ideals of $H_2(D, D_0, \pi, {}^-)$ are exactly the subspaces*

$$H_2(B, B_0) = B_0[11] \oplus B[12] \oplus B_0[22]$$

for $(\pi, {}^-)$-invariant submodules $B_0 \subset D_0$ and $B \subset D$ such that for $d \in D$, $d_0 \in D_0$, $b \in B$, and $b_0 \in B_0$,

(1) *$b\,d + b^\pi\, d^\pi$, $b\, d_0 b^\pi$, and $db_0\, d^\pi$ lie in B_0,*

(2) *db_0, $d_0 b$, dbd, and bdb lie in B.*

In particular, every ideal of $(D, \pi, {}^-)$ gives rise to an ideal of $H_2(D, D_0, \pi, {}^-)$:

(3) $B \lhd (D, \pi, {}^-) \Rightarrow H_2(B, B \cap D_0) \lhd H_2(D, D_0, \pi, {}^-)$.

(4) *If $(D, \pi, {}^-)$ is semiprime, then any nonzero ideal $H_2(B, B_0)$ contains an ideal $H_2(A, A_0)$ for a nonzero $A \lhd (D, \pi, {}^-)$.*

Consequently

(5) $H_2(D, D_0, \pi, {}^-)$ *simple* $\Leftrightarrow (D, \pi, {}^-)$ *simple.*

Proof. Every ideal I splits relative to (e_1, e_2): $I = I_{11} \oplus I_{12} \oplus I_{22}$. Since $I_{ii} = P(e_i) I_{ii} = P(1[12]) I_{jj}$ we may assume

$$I = H_2(B, B_0) = B_0[11] \oplus B[12] \oplus B_0[22]$$

for submodules $B_0 \subset D_0$ and $B \subset D$. A systematic evaluation of the conditions $P(H_2)I + \{H_2 H_2 I\} + P(I) H_2 \subset I$ using (2.2) shows that $H_2(B, B_0)$

is an ideal of $H_2(D, D_0)$ iff for $c, d \in D$, $c_0, d_0 \in D_0$, $a, b \in B$, and $a_0, b_0 \in B_0$ the following expressions lie

in B_0: $d_0\bar{b}_0\, d_0$, $b_0\, \bar{d}_0 b_0$, $c_0\, \bar{d}_0 b_0 + b_0\, \bar{d}_0 c_0$, $b\, \bar{d}_0 b^\pi$, $d\bar{b}_0\, d^\pi$, $d\bar{d}_0 b^\pi + b\, \bar{d}_0\, d^\pi$, $b\, dd_0 + d_0\, d^\pi b^\pi$, $db\, d_0 + d_0 b^\pi\, d^\pi$, $dcb_0 + b_0 c^\pi\, d^\pi$

in B: $db^*\, d$, bd^*b, $cd^*b + bd^*c^*$, $b_0\, \bar{d}_0\, d$, $d_0\bar{b}_0\, d$, $d_0 c_0 b$, $b\bar{c}_0\, d_0$, $d\bar{b}_0\, d_0$, $d\, \bar{d}_0 b_0$, $b_0\, \bar{d}d_0$, $d_0\bar{b}c_0$, $d_0\, \bar{d}b_0$.

These conditions easily reduce to (1), (2), respectively (note that $db\, d \in B$ forces $dbc + cdb \in B$ hence $d \circ b = db + bd \in B$ and thus also $\{cdb\} = c \circ (d \circ b) - \{cbd\} \in B$). Now (3) immediately follows using the ampleness of D_0.

The direction $\Rightarrow$ of (5) follows trivially from (3) and the direction $\Leftarrow$ from (4): if $0 \neq H_2(B, B_0) \lhd H_2(D, D_0, \pi, ^-)$ then $B \supset A = D$ by simplicity of $(D, \pi, ^-)$, so $B = D$. But then also $B_0 = D_0$ (because $bD_0 b^\pi \subset B_0$), so $H_2(B, B_0) = H_2(D, D_0)$.

For (4) let $0 \neq H_2(B, B_0) \lhd H_2(D, D_0, \pi, ^-)$. We claim that there exist b, $b_1 \in B$ with $bb_1 b \neq 0$. Indeed, otherwise $bBb = 0$ for all $b \in B$, thus $bdb\, Dbdb = 0$ by (2) for all $b \in B$ and $d \in D$, saying that $b\, db$ is an absolute zero divisor of the semiprime D, so $b\, db = 0$ for all $d \in D$. But then b is an absolute zero divisor, thus $b = 0$ for all $b \in B$, a contradiction. Therefore, for a suitable choice of b, $b_1 \in B$ we have $0 \neq I = Dbb_1 b\, D$ is an ideal of the associative algebra D contained in B: $d_1 bb_1 b\, d_2 = \{d_1 b, b_1, b\, d_2\} - b(d_2 b_1\, d_1)b \in B$. Then $0 \neq A = I + \bar{I} + I^\pi + \bar{I}^\pi \lhd (D, \pi, ^-)$ is contained in B, so $H_2(A, A_0) \subset B$ for $A_0 = A \cap B_0$. ▮

The description of $(\pi, ^-)$-simple algebras is twice as long as that of $*$-simple algebras.

2.8. PROPOSITION. *For commuting involution π and involutory automorphism $^-$, the $(\pi, ^-)$-simple associative algebras $(C, \pi, ^-)$ are precisely all*

(I) *simple C with internal involution π and automorphism $^-$;*

(II) *π-simple $C = D \boxplus D^\pi \cong D \boxplus D^{\mathrm{op}}$ with exchange involution π for simple D with internal involutory automorphism $^-$ $((d, b)^\pi = (b, d)$, $\overline{(d, b)} = (\bar{d}, \bar{b}))$;*

(III) *π-simple $C = D \boxplus D^\pi = D \boxplus D^{\mathrm{op}}$ with exchange involution π for simple D with internal involution ι $((d, b)^\pi = (b, d)$, $\overline{(d, b)} = (b^\iota, d^\iota))$;*

(IV) *non-π-simple, $^-$-simple $C = B \boxplus \bar{B} \cong B \boxplus B$ with exchange automorphism $^-$ for simple B with internal involution π $((d, b)^\pi = (d^\pi, b^\pi)$, $\overline{(d, b)} = (b, d))$;*

(V) *non-π-, non-$^{-}$-simple* $C = A \boxplus A^{\pi} \boxplus \bar{A} \boxplus \bar{A}^{\pi} = D \boxplus D^{\pi}$ $(D = A \boxplus \bar{A})$ $= B \boxplus \bar{B}$ $(B = A \boxplus A^{\pi})$ *for simple* A *with exchange involution* π *and automorphism* $^{-}$ $((a_1, a_2, a_3, a_4)^{\pi} = (a_2, a_1, a_4, a_3)$, $\overline{(a_1, a_2, a_3, a_4)} = (a_3, a_4, a_1, a_2))$.

Proof. If C is simple we have (I); if C is not simple but is π-simple it is well known that we have $C = D \boxplus D^{\pi}$; the automorphism $^{-}$ can only fix the unique ideals D, D^{π} or switch them, where in the first case we have (II) and in the second (III) since $\overline{(b, c)} = (g(c), f(b))$ of period 2 forces $g = f^{-1}$, $^{-} \circ \pi$ an involution and $\overline{(b, 0)}^{\pi} = (f(b), 0)$ force f to be an involution. If C is not π-simple let B be a proper π-ideal; then (because π and $^{-}$ commute) $B + \bar{B} \neq 0$ and $B \cap \bar{B} \neq C$ are $(\pi, {}^{-})$-ideals, hence $B + \bar{B} = C$, $B \cap \bar{B} = 0$, $C = B \boxplus \bar{B}$. Here B is π-simple since if B_0 is a proper π-ideal of B (hence proper in C) we have as before $C = B_0 \boxplus \bar{B}_0$, so $B_0 = B$. Once more either B is simple, and we have (IV), or $B = A \boxplus A^{\pi} \cong A \boxplus A^{\mathrm{op}}$ for simple A with exchange involution, and we have (V). ∎

2.9. *Remark.* For any any algebra $(C, \pi, {}^{-})$, the symmetric elements $H(C, \pi) = \{c \in C; c^{\pi} = c\}$ are the maximal ample subspace. If π is the exchange involution for $C = D \boxplus D^{\mathrm{op}}$, then $H(C, \pi)$ is the only ample subspace

$$C_0 = H(C, \text{exchange}) \tag{1}$$

since then all symmetric elements $c = c^{\pi}$ are traces $c = d + d^{\pi}$ in C_0. In this case, let $\langle C_0 \rangle \subset C$ be the subalgebra generated by $C_0 = H(C)$. Then

$$D^{(1)} \boxplus D^{\mathrm{op}(1)} \subset \langle C_0 \rangle \qquad (D^{(1)} = D[D, D]\, D). \tag{2}$$

Indeed, since $H(C) = \{(d, d); d \in D\}$ we have $(d_1, d_1)(d_2, d_2) - (d_2 d_1, d_2 d_1) = ([d_1, d_2], 0) \in \langle C_0 \rangle$, so also $(d_1, d_1)([D, D], 0)(d_2, d_2) = (d_1[D, D]\, d_2, 0) \in D^{(1)} \boxplus 0 \subset \langle C_0 \rangle$. Now (2) follows by π-invariance of $\langle C_0 \rangle$. As a corollary,

(3) $H_2(C, C_0, \text{exchange}, {}^{-})$ is diagonal (i.e., $\langle C_0 \rangle = C$) iff $D = D^{(1)}$ (e.g., D simple noncommutative).

Sufficiency of $D = D^{(1)}$ for diagonality is immediate from (2). On the other hand, if we factor out $C^{(1)} = D^{(1)} \boxplus D^{\mathrm{op}(1)}$, the image of $\langle C_0 \rangle$ in the commutative quotient $C/C^{(1)}$ is just the symmetric elements $\{(d, d); d \in D/D^{(1)}\}$ forcing $C = C^{(1)}$, i.e., $D = D^{(1)}$ by diagonality. ∎

2.10. Hermitian Simplicity Criterion. *A hermitian matrix system is simple iff it is isomorphic to one of the following*:

(I) *$H_2(D, D_0, \pi, {}^-)$ for simple D with π, ${}^-$;*

(II) *$M_2(D, {}^-)$ for a simple D with automorphism ${}^-$, $P(x)\,y = x\bar{y}x$, and $\overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}$;*

(III) *$M_2(D, \iota)$ for a simple D with involution ι, $P(x)\,y = x\bar{y}'x$, and $\overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$;*

(IV) *polarized $(H_2(B, B_0, \pi),\ H_2(B, B_0, \pi))$ for a simple B with involution π;*

(V) *polarized $(M_2(A), M_2(A))$ for a simple A.*

Among the cases (II)–(V), *the matrix system is diagonal iff D, B, A resp. are noncommutative.*

Proof. The result follows from the previous propositions and the remark. Note that always $H_2(D \boxplus D^{\mathrm{op}}, D, \text{exchange}, {}^-) \cong M_2(D)$ with $P(x)\,y = x\bar{y}x$ or $x\bar{y}'x$ depending on whether ${}^-$ is internal or not (since under $M_2(D \boxplus D^{\mathrm{op}}) \to^{1 \oplus t} M_2(D) \boxplus M_2(D)^{\mathrm{op}}$ the exchange becomes $(x, y)^\pi = (y', x')$ and H_2 becomes $\{(x, x')\}$). Also, $H_2(B \boxplus B, B_0 \boxplus B_0, \pi, \text{exchange}) = (H_2(B, B_0, \pi), H_2(B, B_0, \pi))$ viewed as polarized Jordan triple system. ∎

Since a hermitian matrix system $H_2 = H_2(D, D_0, \pi, {}^-)$ is built out of $(D, D_0, \pi, {}^-)$, algebraic properties of H_2 can be characterized in terms of the coordinate system $(D, D_0, \pi, {}^-)$—see, for example, (2.7.5). Other examples are given in

2.11. Lemma. *Semiprimeness* (*no trivial ideals*) *and primeness* (*semiprimeness plus finite intersection property for ideals*) *are hereditary*:

(a) *H_2 is nondegenerate $\Leftrightarrow$ H_2 is semiprime $\Leftrightarrow (D, \pi, {}^-)$ is semiprime $\Leftrightarrow D$ is semiprime.*

(b) *H_2 is prime $\Leftrightarrow (D, \pi, {}^-)$ is prime.*

Proof. (a) H_2 nondegenerate implies H_2 semiprime which implies $(D, \pi, {}^-)$ semiprime by (2.2) and (2.7.3), and this forces D itself to be semiprime (if $I \lhd D$ has $I^2 = 0$, then $\hat{I} = I + \bar{I} + I^\pi + \bar{I}^\pi \lhd (D, \pi, {}^-)$ has $\hat{I}^5 < 0$, so $I \subset \hat{I} = 0$). Finally, assume D semiprime and let $z \in H_2(B, B_0)$ be an absolute zero divisor, $zH_2z = 0$. Then the components of z in J_1, J_{12}, J_2 are also absolute zero divisors. By (2.2) the J_{ii}-component $d_0[ii]$ of z has $d_0\, D_0\, d_0 = 0$. Then $d_0(d + d^\pi)\, d_0 = d_0(d\, d_0\, d^\pi)\, d_0 = 0$ shows any $d_0\, d\, d_0$ is an absolute zero divisor, $(d_0\, d\, d_0)\, c(d_0\, d\, d_0) = -d_0\, d\, d_0(d^\pi\, d_0 c^\pi)\, d_0 = 0$, so by nondegeneracy of D all $d_0\, d\, d_0 = 0$ and hence $d_0 = 0$. But then $z = d[12]$ has $d\bar{c}^\pi\, d = 0$ for all $c \in D$ by (2.2) again, hence d is an absolute zero divisor, thus $d = 0$ and $z = 0$.

(b) Since every ideal of $(D, \pi, {}^-)$ generates an ideal of $H_2(D, D_0, \pi, {}^-)$ by (2.7.3), the finite intersection property holds for

$(D, \pi, {}^-)$ if it holds for $H_2(D, D_0, \pi, {}^-)$. Conversely, let $H_2(B_i, B_{0i}) \lhd H_2(D, D_0, \pi, {}^-)$, $i = 1, 2$, with zero intersection. Then $B_1 \cap B_2 = 0$. We may assume $B_1 \neq 0$ and $B_2 \neq 0$. Then there exist by (2.7.4) nonzero ideals $0 \neq A_i \lhd (D, \pi, {}^-)$ with $A_i \subset B_i$. But $A_1 \cap A_2 \subset B_1 \cap B_2 = 0$, so $A_1 = 0$ or $A_2 = 0$ by the finite intersection property of $(D, \pi, {}^-)$, a contradiction. ∎

3. Clifford Coordinatization

In this section we show how to coordinatize triple systems coming from quadratic forms. In general, if C is a commutative associative k-algebra with involution $c \to \bar{c}$, V a C-module, q: $V \to C$ a C-quadratic form, and S: $V \to V$ an involutory hermitian isometry

$$S(cx) = \bar{c}S(x), \qquad q(S(x)) = \overline{q(x)}, \qquad S^2 = I,$$

then we get a *full Clifford system* (sometimes also called a *quadratic form triple*)

$$(3.1) \qquad J(q, S): \qquad P(x)\,y = q(x, Sy)x - q(x)\,S(y).$$

This is actually K-linear for $K = H(C, {}^-)$, the subalgebra of symmetric elements of C (and is an "antilinear" triple over C itself). Clearly there is no loss of information in assuming V is faithful as a C-module (replacing C by $C' = C/\mathrm{Ann}\ V$).

It is easy to see that $J(q, S) = J(q', S')$ if $q' = \lambda q$, $S' = \bar{\lambda}S$ for a unitary $\lambda \in C$ $(\lambda\bar{\lambda} = 1)$, so we may always adjust by unitary scalars. The following result is easy over fields, but actually holds for any C.

3.2. Clifford Triangle Theorem. *(e_1, u, e_2) is a triangle for a full Clifford system $J(q, S)$ iff (q, S) can be normalized so that*

(3.2.1) *e_1, e_2 is a hyperbolic pair orthogonal to a unitary u:* $q(e_1) = q(e_2) = q(e_1, u) = q(e_2, u) = 0$, $q(u) = -q(e_1, e_2) = 1$;

(3.2.2) *S fixes u and negatively switches e_1 and e_2:* $S(e_1) = -e_2$, $S(e_2) = -e_1$, $S(u) = u$.

Proof. It is straightforward to show that if (3.2.1)–(3.2.2) hold then (e_1, u, e_2) is a triangle in the sense of (1.5). From now on we assume (e_1, u, e_2) is a triangle and struggle to establish (3.2.1)–(3.2.2). Direct calculation from (3.1) shows

$$(3.2.3) \qquad q(P(x)\,y) = q(x)^2\,\overline{q(y)}.$$

Further,

$$\text{(3.2.4)} \qquad x \text{ invertible in } J \Rightarrow q(x) \text{ invertible in } C$$

since $J = P(x)J \subset q(x, J)x - q(x)J$ shows $q(x, J) \subset q(x, Cx + q(x)J) \subset Cq(x)$ so $J \subset q(x)J$; then $x = P(x)y \Rightarrow q(x) = q(x)^2\,\overline{q(y)}$ (by 3.2.3) $\Rightarrow q(x)\,\overline{q(y)} = 1$ on $q(x)J = J$, hence by faithfulness $q(x)\,\overline{q(y)} = 1$ in C and $q(x)$ is invertible in C. In particular, $q(e) = \varepsilon$ is invertible. From $\varepsilon = q(e) = q(P(e)e) = \varepsilon^2\bar{\varepsilon}$ we see $\varepsilon\bar{\varepsilon} = 1$, ε is unitary, hence we may scale by $\lambda = -\bar{\varepsilon}$ to get $q'(e) = -\bar{\varepsilon}q(e) = -\bar{\varepsilon}\varepsilon = -1$, i.e., we may replace (q, S) by (q', S') so that our normalized system satisfies

$$\text{(3.2.1a)} \qquad q(e) = -1.$$

If we set $\varepsilon_i = q(e_i)$ then $0 = q(P(e_i)e_j) = \varepsilon_i^2\bar{\varepsilon}_j$, $\varepsilon_j = q(P(e)e_j) = (-1)^2\,\bar{\varepsilon}_j = \bar{\varepsilon}_j$, so $\varepsilon_i = q(P(e_i)e_i) = \varepsilon_i^2\bar{\varepsilon}_i = \varepsilon_i^3$, but $e = P(u)e$ shows $-1 = q(u)^2(-1)$, i.e., $q(u)^2 = 1$, so $\varepsilon_i = q(P(u)e_j) = q(u)^2\,\bar{\varepsilon}_j = \bar{\varepsilon}_j$ and $0 = \varepsilon_i^2\bar{\varepsilon}_j = \varepsilon_i^3 = \varepsilon_i$. Since $-1 = q(e) = \varepsilon_1 + q(e_1, e_2) + \varepsilon_2$ we have

$$\text{(3.2.1b)} \qquad q(e_1) = q(e_2) = 0, \qquad q(e_1, e_2) = -1.$$

Thus $0 = -q(P(e_i)e_j, e_j) = -q(q(e_i, Se_j)e_i, e_j) = q(e_i, Se_j)$ and $1 = -q(e_i, e_j) = -q(P(e_i)e_i, e_j) = -q(q(e_i, Se_i)e_i, e_j) = q(e_i, Se_i)$, so $0 = \{e_i e_i e_j\} = q(e_i, Se_i)e_j + q(e_j, Se_i)e_i - q(e_i, e_j)Se_i = e_j + 0 + Se_i$, hence

$$\text{(3.2.2a)} \qquad Se_i = -e_j.$$

Finally, we note that $u = \{e_1 u e_2\} = q(e_1, Su)e_2 + q(e_2, Su)e_1 - q(e_1, e_2)\,Su$ shows by equating components in Peirce spaces that $u = Su$ and $0 = q(e_i, Su)e_j = q(e_i, u)e_j$, hence applying $-q(\cdot, e_i)$ as usual gives $q(e_i, u) = 0$,

$$\text{(3.2.2b)} \qquad Su = u,$$

$$\text{(3.2.1c)} \qquad q(e_i, u) = 0.$$

Then $1 = -q(e_1, e_2) = -q(P(u)e_2, e_2) = -q(q(u, Se_2)u - q(u)\,Se_2, e_2) = -0 + q(u)\,q(Se_2, e_2)$ (by 3.2.1c) $= -q(u)\,q(e_1, e_2)$ (by 3.2.2a) $= q(u)$ (by 3.2.1b), establishing

$$\text{(3.2.1d)} \qquad q(u) = 1.$$

This shows that the inner products (3.2.1)–(3.2.2) are necessary consequences of triangularity. ∎

Note that (3.2) implies that C acts faithfully on V, indeed on e_1:

$$ce_1 = 0 \Rightarrow c = -cq(e_1, e_2) = q(ce_1, e_2) = 0.$$

In the triangulated case we get a Peirce decomposition of the module V:

(3.3.i) $$V = Ce_1 \oplus M \oplus Ce_2$$

(3.3.ii) $q(c_1e_1 \oplus m \oplus c_2e_2) = q(m) - c_1c_2, \qquad q: M \to C$

(3.3.iii) $S(c_1e_1 \oplus m \oplus c_2e_2) = (-\bar{c}_2e_1) \oplus S(m) \oplus (-\bar{c}_1e_2), \quad S: M \to M$

(3.3.iv) $P(c_1e_1 + m + c_2e_2)(b_1e_1 + n + b_2e_2) = d_1e_1 \oplus p \oplus d_2e_2$

$d_i = c_i^2\bar{b}_i + c_iq(m, S(n)) + \bar{b}_jq(m)$

$p = [c_1\bar{b}_1 + c_2\bar{b}_2 + q(m, S(n))]m + [c_1c_2 - q(m)]\, S(n)$

(3.3.v) $\{c_1e_1 + m + c_2e_2, b_1e_1 + n + b_2e_2, c_1'e_1 + m' + c_2'e_2\}$

$= d_1e_1 \oplus p \oplus d_2e_2$

$d_i = q(c_im' + c_i'm, S(n)) + b_jq(m, m') + 2c_ic_i'\bar{b}_i$

$p = [c_1\bar{b}_1 + c_2\bar{b}_2 + q(m, S(n)]\, m'$

$+ [c_1'\bar{b}_1 + c_2'\bar{b}_2 + q(m', S(n))]m$

$+ [c_1c_2' + c_1'c_2 - q(m, m')]\, S(n).$

Our derived operations of Section 1 are

(3.4.i) $\overline{ce_i} = \bar{c}e_i, \quad (ce_i)^* = ce_j \quad (ce_i)^\mu = \bar{c}e_j$

(3.4.ii) $\bar{m} = S(m), \qquad m^* = T(m)u - m, \qquad Q_j(m) = q(m)e_j,$
$T_i(m) = q(m, u)e_i$

(3.4.iii) $ce_i \cdot m = (ce_i)^* \cdot m = cm,$
$P(m)(ce_i)^\mu \cdot n = Q_1(m) \cdot (ce_1 \cdot n) = q(m)\, cn.$

In general we need not take the full Peirce spaces Ce_i in order to get triangulated triples.

3.5. PROPOSITION. *The triple subsystems of a full triangulated Clifford system $J(q, S)$ containing (e_1, e_2, u) are precisely all systems*

$$J_0 = J(q, S, C_0) = C_0e_1 \oplus M_0 \oplus C_0e_2,$$

where $C_0 \subset C$ and $M_0 \subset M$ satisfy

(i) $1 \in C_0$, $\bar{C}_0 = C_0$,

(ii) $u \in M_0$, $S(M_0) = M_0$,

(iii) $C_0M_0 \subset M_0$, $C_0\, q(M_0) \subset C_0 \qquad$ (*so* $C_0 \cdot C_0^2 \subset C_0$).

Such J_0 *will be an outer ideal of* J $[P(J)J_0+\{JJJ_0\}\subset J_0]$ *iff* $M_0=M$, *so* (i)–(iii) *reduce to*

(i)′ $1\in C_0$, $C_0=\bar{C}_0$,

(iii)′ $C_0q(M)\subset C_0$.

When J_0 *is an outer ideal,*

(iv) u *is faithful iff* $\mathrm{Ann}_{C_0}(u)=0$ $(c_0u=0\Rightarrow c_0=0$ *for* $c_0\in C_0)$;

(v) u *is C-faithful iff* $\mathrm{Ann}_{k[C_0]}(u)=0$ $(cu=0\Rightarrow c=0$ *for* c *in the* k*-subalgebra* $k[C_0]\subset C$ *generated by* $C_0)$; *thus we have C-faithfulness if* C *has no* 2*-torsion or no nilpotents or if* $1\in q(u,M)$;

(vi) *if* $\frac{1}{2}\in k$ *or, more generally, if* $1\in q(M,M)$, *then* $C_0=C$ *and* $J_0=J$ *is full.*

Proof. If e_1, $e_2\in J_0$ then the Peirce decomposition yields $J_0=C_1e_1\oplus M_0\oplus C_2e_2$ for k-subspaces $C_i\subset C$ with $1\in C_i$; here $u\in J_0\Leftrightarrow u\in M_0$, so closure under $P(e_1,e_2)$ shows $SM_0=M_0$, closure under $P(u)$ with $q(u)=1$ shows $\bar{C}_i\subset C_j$, $C_j=\bar{C}_1$, and closure under $P(e_1)$ shows $\bar{C}_1=C_1$, so $C_1=C_2=:C_0=\bar{C}_0$. Then closure under $P(x_0)y_0$ reduces to (iii). If J_0 is outer, then $M=\{e_1e_1M\}\subset\{J_0J_0J\}\subset J_0$ shows $M=M_0$. Conversely, $M=M_0$ and (i′), (iii′) imply $P(J)J_0+\{JJJ_0\}\subset J_0$ by the multiplication rules (3.3).

The faithfulness criteria (iv), (v) result from (3.4)(iii). If $cu=0$ then $0=q(cu)=c^2q(u)=c^2$ and $0=q(cu,M)=cq(u,M)$ (so $0=cq(u,u)=2c$): $A=\mathrm{Ann}_C(u)\lhd C$ has $2A=A^2=Aq(u,M)=0$, so vanishes if C has no 2-torsion, no nilpotents, or if $1\in q(u,M)$.

If $\frac{1}{2}\in k$ then $1=\frac{1}{2}q(u,u)\in q(M,M)$, and always $Cq(M,M)=q(M,CM)=q(M,M)\subset C_0$. ∎

Such outer ideals will be called *ample Clifford systems* (or ample quadratic form triples):

$$J(q,S,C_0)=C_0\oplus M\oplus C_0,$$

where $C_0\subset C$ is an ample subspace $(1\in C_0=\bar{C}_0,\ C_0q(M)\subset C_0)$ of the commutative associative k-algebra with involution $c\to\bar{c}$, M is a C-module, q: $M\to C$ is a C-quadratic form, and S: $M\to M$ is an involutory hermitian isometry. The product of $J(q,S,C_0)$ is given by (3.1), where S and q are extended to $V=C\oplus M\oplus C$ by (3.3). As (2.5)(vi) shows, ample Clifford systems are full in characteristic $\neq 2$.

In the recent characterization of prime Jordan algebras [8], a key result was the existence of Zel'manov polynomials which distinguish between hermitian matrix algebras $H_0(A,*)$ and Clifford forms. In this spirit the following criterion characterizes ample Clifford systems by a polynomial identity:

3.6. CLIFFORD CRITERION. *A triangulated* $J = J_{11} \oplus M \oplus J_{22}$ *is isomorphic to an ample Clifford system* $J(q, S, C_0)$ *iff* u *is faithful and*

(C) $$(P(m)x_1^\mu) \cdot n = Q_1(m) \cdot (x_1 \cdot n)$$

holds for all $x_1 \in J_{11}$ *and* $m, n \in M$. *More precisely, faithfulness and* (C) *imply*

(3.6.1) *the subalgebra* C *of* End M *generated by* $C_0 = \{L(x_1);\ x_1 \in J_{11}\}$ *is a commutative associative* k*-algebra with involution,*

(3.6.2) $q\colon M \to C\colon m \to L(Q_1(m))$ *is a* C*-quadratic form,*

(3.6.3) $S\colon M \to M\colon m \to \bar{m}$ *is an involutory antilinear isometry,*

(3.6.4) $C_0 \subset C$ *is ample,*

(3.6.5) *the map* $\varphi\colon J \to J(q, S, C_0)\colon v = x_1 \oplus m \oplus y_1^* \to L(x_1) \oplus m \oplus L(y_1)$ *is an isomorphism.*

In short, a faithfully triangulated J is a Clifford system iff $\Delta_1(J_1; M) = 0$, where

$$\Delta_i(x_i; m) = L(P(m)\, x_i^\mu) - L(Q_i(m))\, L(x_i).$$

A linearized form of Δ_i is

$$\Gamma_i(x_i; m) = L(T_i(x_i \cdot m)) - L(T_i(m))\, L(x_i),$$

indeed, we always have

(3.7) $$\Delta_i(x_i; m, u) = \Gamma_i(x_i; m^*)$$

(since $\Gamma_i(x_i; m^*) = L(T_i((x_i^* \cdot m)^*)) - L(T_i(m^*))\, L(x_i) = L(T_i(x_i^* \cdot m)) - L(T_i(m))\, L(x_i)$ (by (1.6.2)) $= L(\{m x_i^\mu u\}) - L(Q_i(m, u))\, L(x_i)$ (by (1.3.4)) $= \Delta(x_i; m, u)$), so in particular

$$\Delta_i \equiv 0 \Rightarrow \Gamma_i \equiv 0.$$

Also,

$$\Gamma_i \equiv 0 \Rightarrow (x_i - x_i^*) \cdot m = 0$$

because $(x_i - x_i^*) \cdot m = \Gamma_i(x_i; m) \cdot u$ by (1.6.2).

Before we give the proof of (3.6) we will study the converse of the implications $\Delta_i \equiv 0 \Rightarrow \Gamma_i \equiv 0 \Rightarrow (x_i - x_i^*)\cdot m = 0$ (this will be needed in Section 4) and derive some general formulas used in the proof of (3.6).

3.8. ($\Gamma \Rightarrow \Delta$) LEMMA. *If either*

(i) $\Gamma_1(J_1; M) \equiv 0$

or

(i′a) *u is faithful, and*

(i′b) $(x_1 - x_1^*)\cdot m \equiv 0$

hold, then $\Delta_i(J_i; M) \equiv 0$ *if any one of the following cases hold*:

(I) $\frac{1}{2} \in k$,

(II) *u is C-faithful.*

Proof. Since $\Gamma_1(x_1; m)^* = \Gamma_2(x_i^*; m^*)$ we have (i) $\Rightarrow \Gamma_i(J_i; M) = 0$ and similarly (i′b) $\Rightarrow (x_i - x_i^*)\cdot m$. We already noted that (i) $\Rightarrow$ (i′b), so we first draw some consequences of (i′b) alone:

(1) $L(x_i) = L(x_i^*)$,

(2) C is commutative,

(3) $L(T_i(m)) = L(T_j(m)) = L(T_i(m^*))$,

(4) $L(Q_i(x_i \cdot m, n)) = L(Q_i(m, x_i \cdot n))$,

and a consequence of (i) or (i′):

(5) $L(Q_i(m)) = L(Q_j(m)) = L(Q_i(m^*))$.

Indeed, (1) follows from (i′b); (2) holds since by (1.3.6), $L(J_1)$ commutes with $L(J_2)$ so all generators $L(x_1)$, $L(y_1) = L(y_1^*)$ of C commute; (3) is just (1.6)(2) and (1); by (1.3.4), (4) holds for x_j in place of x_i, hence holds also for x_i by $L(x_i) = L(x_i^*) \in L(J_i)$; when $\Gamma \equiv 0$ then (5) follows from the general formula

$$\Gamma_i(T_i(m); m) = L(Q_i(m^*) - Q_i(m)), \tag{3.9}$$

which holds because $Q_i(m^*) - Q_i(m) = Q_i(T_i(m)\cdot u - m) - Q_i(m) = P(T_i(m))\,Q_i(u) - Q_i(T_i(m)\cdot u, m)$ (by (1.3.8)) $= P(T_i(m))e_i + T_i(T_i(m)\cdot m) - \{T_i(m)\, e_i T_i(m)\}$ (by linearized (1.3.9)) $= -P(T_i(m))e_i + T_i(T_i(m)\cdot m)$, therefore $L(Q_i(m^*) - Q_i(m)) = -L(T_i(m))^2 + L(T_i(T_i(m)\cdot m)) = \Gamma_i(T_i(m); m)$. When (i′) holds, then by (1.6.3), (4)

$$\begin{aligned}(Q_i(m^*) - Q_i(m))\cdot u &= \Gamma_i(T_i(m); m)\cdot u\\ &= (T_i(m) - T_i(m)^*)\cdot m = 0,\end{aligned}$$

so (5) follows by faithfulness. Under conditions (i) or (i′) we also know

(6) $\Delta_i(x_i; m)n = Q_i(n, x_i \cdot m) \cdot m - Q_i(n, m) \cdot (x_i \cdot m)$ since $\Delta_i n = P(m)x_i^\mu \cdot n - Q_i(m) \cdot (x_i \cdot n) = (-\{P(m)e_i, x_i^\mu, n\} + \{m\{e_i m x_i^\mu\}n\}) - Q_i(m) \cdot (x_i \cdot n) = -Q_i(m) \cdot \underline{(x_i^* \cdot n)} + P(m, n)(x_i^\mu \cdot \bar{m}) - Q_i(m) \cdot (x_i \cdot n) = -2Q_i(m) \cdot (x_i \cdot n) + P(m, n)\overline{(x_i^* \cdot m)}$ (by (5), (1)) $= -[L(x_i)\, L(Q_i(m)) + L(Q_i(m))\, L(x_i)]n + Q_i(m, x_i \cdot m) \cdot n + Q_i(n, x_i \cdot m) \cdot m - Q_i(m, n) \cdot (x_i \cdot m)$ (by (2), (1.3.2), (1)) $= L(Q_i(x_i \cdot m, m) - \{x_i e_i Q_i(m)\})n + Q_i(n, x_i \cdot m) \cdot m - Q_i(n, m) \cdot (x_i \cdot m)$ (by (1.3.9), (5)) $= Q_i(n, x_i \cdot m) \cdot m - Q_i(n, m) \cdot (x_i \cdot m)$ (by (1.3.9)).

We now get (I) immediately: $2\,\Delta_i n = L(Q_i(x_i \cdot n, m) + Q_i(n, x_i \cdot m))m - [L(Q_i(n, m))\, L(x_i) + L(x_i)\, L(Q_i(n, m))]m$ (by (6), (4), (2)) $= 0$ by (1.3.9) and (1.3.7). We also get (II): $\Delta_i u = T_i(x_i \cdot m) \cdot m - T_i(m) \cdot (x_i \cdot m)$ (by (6)) $= \Gamma_i(x_i; m) \cdot m$ and $\Gamma_i(x_i; m) \cdot u = (x_i - x_i^*) \cdot m = 0$, so C-faithfulness $cu = 0 \Rightarrow c = 0$ yields $\Gamma_i \equiv 0$ and then $\Delta_i \equiv 0$. ▌

Remark. Under conditions (i) or (i′) (and without (I) or (II)) we have

$$\Delta_i(P(x_i)\, y_i; m) = L(x_i)^2\, \Delta(\bar{y}_i; m),$$

so $\Delta_i(x_i^2; m) = 0$ for $x_i^2 = P(x_i)e_i$, since by (6) and (1.3.7), $\Delta_i(P(x_i)\, y_i; m)n = Q_i(n, x_i \cdot (\bar{y}_i \cdot (x_i \cdot m))) \cdot m - Q_i(n, m) \cdot (x_i \cdot (\bar{y}_i \cdot (x_i \cdot m))) = Q_i(x_i \cdot n, x_i \cdot (\bar{y}_i \cdot m)) \cdot m - L(x_i)^2\, Q_i(n, m) \cdot (\bar{y}_i \cdot m)$ (by (1.3.4), (4), and (2)) $= (P(x_i)\, \overline{Q_i(n, \bar{y}_i \cdot m)}) \cdot m + L(x_i)^2\, \Delta_i(\bar{y}_i; m)n - L(x_i^2)\, Q_i(n, \bar{y}_i \cdot m) \cdot m$ (by (1.3.8) and (6)) $= L(x_i)^2\, \Delta_i(\bar{y}_i; m)n$. Note that linearizing $\Delta_i(x_i^2; m) = 0$ gives $\Delta_i(2x_i; m) = 2\Delta_i(x_i; m) = 0$. ▌

Remark. Under conditions (i) or (i′) one always has $\Delta_i(x_i; m)\, Cm = 0$, since $\Delta_i(x_i; m)\, m = Q_i(m, x_i \cdot m) \cdot m - 2Q_i(m) \cdot (x_i \cdot m)$ (by (6)) $= \{x_i e_i Q_i(m)\} \cdot m - 2Q_i(m) \cdot (x_i \cdot m))$ (by (1.3.9)) $= 0$ by (1.3.7) and commutativity of C. ▌

Proof of 3.6. By (3.4) the Clifford condition (C) certainly is necessary. So assume J is faithfully triangulated and satisfies (C). Then, as noted before, $\Gamma_1 \equiv 0$, and $L(x_1) = L(x_1^*)$ for all $x_1 \in J_1$, and therefore all the consequences (1)–(6) of (3.8) hold too. In particular, C is commutative.

We have a quadratic form $Q_1\colon M \to J_{11}$, and a specialization $J_{11} \xrightarrow{L} C$, so composing yields a k-quadratic form

$$q\colon M \to C \qquad \text{by} \quad q(m) = L(Q_1(m)).$$

This is actually C-quadratic: it suffices to show $q(cm) = c^2 q(m)$, $q(cm, n) = cq(m, n)$ for generators $c = L(x_1)$ of C; but $q(L(x_1)m) = L(Q_1(x_1 m)) = L(P(x_1)\, \overline{Q_1(m)})$ (by (1.3.8)) $= L(x_1) L(Q_1(m)) L(x_1) = L(x_1)^2 q(m)$ (by commutativity of C) and $q(L(x_1)m, n) = L(Q_1(x_1 \cdot m, n)) = L(Q_1(x_i^* \cdot m, n)) = L(\{m \bar{x}_1^* n\})$ (by (1.3.4)) $= L(P(m, n)x_1^\mu) = L(Q_1(m, n))\, L(x_1)$ (by linearized (C)) $= L(x_1)\, q(m, n)$ (by commutativity of C).

The map $S(m)=\bar{m}$ is clearly involutory. It is antilinear: $S(x_1\cdot m)=P_e(x_1\cdot m) = P_eL(x_1)P_eSm=\overline{L(x_1)}\,Sm$ implies $S(cm)=\bar{c}S(m)$ for $c\in C$. It is also an isometry: $q(Sm)=L(Q_1(\bar{m}))=L(\overline{Q_1(m)})$ (by (1.3.12)) $=P_eL(Q_1(m))P_e=\overline{q(m)}$. Finally, $C_0=L(J_{11})\subset C$ is ample in the sense of 3.5: $1=L(e_1)\in C_0$, $\overline{L(x_1)}=L(\bar{x}_1)\in C_0$, and $q(m)\,L(x_1)=L(P(m)x_1^\mu)\in C_0$ by (C). Thus we can form the ample Clifford system $J(q, S, C_0)$.

The map defined in (3.6.5) is k-linear, injective (by faithfulness $L(x_1)=0\Rightarrow x_1=0$), and maps onto $J(q, S, C_0)$ (by definition of $C_0=L(J_{11})\subset C$); it is a homomorphism by the morphism criterion 1.4 since it preserves Peirce spaces, maps the triangle (u, e_1, e_2) onto the triangle $(u, L(e_1), L(e_2))$, and by (3.3iv) preserves the products

$$\begin{aligned}
\varphi(P(x_1)\,y_1) &= L(P(x_1)\,y_1)\\
&= L(x_1)\,L(\bar{y}_1)\,L(x_1) \qquad \text{(by (1.3.7))}\\
&= L(x_l)^2\,L(\bar{y}_1) \qquad \text{(by commutativity of } C)\\
&= \varphi(x_1)^2\,\overline{\varphi(y_1)}\\
\varphi(P(m)\,y_1^\mu) &= L(P(m)\,y_1^\mu)=L(Q_1(m))\,L(y_1) \quad \text{(by (C))}\\
&= L(y_1)\,q(m)=L_2(y_1^*)\,q(m)\\
&= \overline{L_2(y_1^\mu)}\,q(m)\\
\varphi(x_1\cdot m) &= L(x_1)m=\varphi(x_1)\cdot m. \quad \blacksquare
\end{aligned}$$

Now we are ready to recognize a Clifford part inside any triangulated system.

3.10. CLIFFORD COORDINATIZATION THEOREM. *If J is faithfully triangulated then it contains an ample Clifford subsystem*

$$J_q\cong J(q, S, C_0)$$

defined by $J_q=K_1\oplus N\oplus K_2$, *where*

(i) $N_0=\{n_0\in M;\ \Gamma_i(J_i; n_0)=0$ *for* $i=1, 2\}$,

(ii) $K_i=\{k_i\in J_i;\ \Gamma_i(k_i; C_iN_0)=0\}$ (*where C_i is the subalgebra of* End M *generated by* $L(J_i)$),

(iii) $N=\{n\in M;\ \Delta_i(J_i; n)=\Delta_i(J_i; n, N_0)=\Gamma_i(T_i(n);\ C_iN_0)=0$ *for* $i=1, 2\}$

for operators

$$\Gamma_i(x_i; m)=L(T_i(x_i\cdot m))-L(T_i(m))\,L(x_i)$$

$$\Delta_i(x_i; m)=L(P(m)\,x_i^\mu)-L(Q_i(m))\,L(x_i)$$

in $\mathrm{End}_k(M)$. J_q *always contains the triangle* (e_1, e_2, u) *and coincides with* J *iff* $\Delta_1(J_1; M) \equiv 0$.

Proof. The hard part is proving that J_q is a subsystem; since $\Delta_1(J_1; N) = 0$ acting on M, we certainly have $\Delta_1(K_1; N) = 0$ acting on N, so by the Clifford Criterion (3.6), J_q will be an ample Clifford system.

We have $e_i \in K_i$ and $u \in N_0$, N straight from the definitions: from (1.3.7), (1.3.12), (1.5), and (1.6.2),

$$(1) \qquad \Gamma_i(e_i; M) = \Delta_i(e_i; M) = 0, \qquad \Gamma_i(J_i; u) = \Delta_i(J_i; u) = 0$$

$$(2) \qquad \Delta_i(x_i; m)^* = \Delta_j(x_i^*; m^*), \qquad \overline{\Delta_i(x_i; m)} = \Delta_i(\bar{x}_i; \bar{m})$$

$$(3) \qquad \Gamma_i(x_i; m)^* = \Gamma_j(x_i^*; m^*), \qquad \overline{\Gamma_i(x_i; m)} = \Gamma_i(\bar{x}_i; \bar{m}).$$

Consequently

$$(4) \qquad \bar{N}_0 = N_0 = N_0^*,$$

and because $(C_i \cdot N_0)^* = C_j N_0 = \overline{(C_j \cdot N_0)}$ we have

$$(5) \qquad \bar{K}_i = K_i = K_j^*, \qquad \bar{N} = N = N^*.$$

Moreover,

$$(6) \qquad N \subset N_0$$

since $\Gamma_j(J_j; N) = \Delta_i(J_i; N, u)^*$ (by (3.7) and (2)) $\subset \Delta_i(J_i; N, N_0)^* = 0$. This in particular shows that N is a k-subspace: all defining conditions for N are linear except $\Delta_i(J_i; n) = 0$, however, for $n_1, n_2 \in N$ we have $\Delta_i(J_i; n_1 + n_2) = \Delta_i(J_i; n_1, n_2) \subset \Delta_i(J_i; n_i, N_0) = 0$.

By (1.3) and (5), J_q will be closed under triple products as soon as

$$\text{(a) } K_i N \subset N, \quad \text{(b) } P(K_i) K_i \subset K_i, \quad \text{(c) } P(N) K_j \subset K_i.$$

Before proving (a) we need some more formulas:

$$(7) \qquad [L(K_i), L(J_i)] = 0,$$

which follows from the more general formula

$$(8) \qquad [L(x_i), L(y_i)] = \Gamma_i(x_i; y_i \cdot u) \quad (x_i, y_i \in J_i)$$

(for $\Gamma_i(x_i; y_i \cdot u) = L(T_i(x_i \cdot (y_i \cdot u))) - L(T_i(y_i \cdot u)) L(x_i) = L(\{x_i \bar{y}_i e_i\}) - 2L(y_i) L(x_i)$ (by (1.6.5)) $= L(x_i) L(y_i) - L(y_i) L(x_i)$ by (1.3.7)). We also have in general

$$(9)\qquad \Delta_i(x_i;\, y_i\cdot m)=L(y_i)\,\Delta_i(x_i;\, m)\,L(y_i) + L(y_i)\,L(Q_i(m))[L(x_i),\, L(y_i)]$$

since $L(P(y_i\cdot m)x_i^\mu)-L(Q_i(y_i\cdot m))\,L(x_i) = L(P(y_i)\,\overline{P(m)x_i^\mu})-L(P(y_i)\,\overline{Q_i(m)})\,L(x_i)$ (by (1.3.8)) $=L(y_i)\,L(P(m)x_i^\mu)\,L(y_i)-L(y_i)\,L(Q_i(m))\,L(y_i)\,L(x_i)$ (by (1.3.7)) $=L(y_i)\{[\Delta_i(x_i;m)+L(Q_i(m))\,L(x_i)]\,L(y_i)-L(Q_i(m))\,L(y_i)\,L(x_i)\}$. Thus

$$(10)\qquad \Delta_i(x_i;\, k_i\cdot n)=0 \qquad (k_i\in K_i,\, n\in N)$$

since $\Delta_i(x_i;n)=0$ for $n\in N$ and $[L(x_i),\, L(k_i)]=0$ by (7). Linearizing (9) in m gives $\Delta_i(x_i;\; k_i\cdot n,\; k_i\cdot n_0)=0$ for $k_i\in K_i$, $n\in N$, and $n_0\in N_0$ since $\Delta_i(x_i;n,n_0)=0$, hence a second linearization $k_i\to k_i,\; e_i$ gives $\Delta_i(x_i;k_i\cdot n,n_0)=-\Delta_i(x_i;n,k_i\cdot n_0)$, so

$$(11)\qquad \Delta_i(J_i;\, k_i\cdot n,\, N_0)=0$$

will follow from

$$(12)\qquad K_i\cdot N_0\subset N_0.$$

But in general for $x_i^2=P(x)e_i$,

$$(13)\qquad \Gamma_i(x_i^2;m)-\Gamma_i(x_i;\, x_i\cdot m)=\Gamma_i(x_i;m)\,L(x_i)$$

since $L(T_i(x_i^2\cdot m-x_i\cdot(x_i\cdot m)))-L(T_i(m))\,L(x_i^2)+L(T_i(x_i\cdot m))\,L(x_i) = 0+\Gamma_i(x_i;\, m)\,L(x_i)$ by (1.3.7), so linearizing $x_i\to x_i,\; k_i$ shows $\Gamma_i(x_i;k_i\cdot n_0) = -\Gamma_i(k_i;x_i\cdot n_0) + \Gamma_i(\{k_ie_ix_i\};n_0) - \Gamma_i(x_i;n_0)\,L(k_i) - \Gamma_i(k_i;n_0)\,L(x_i)\in\Gamma_i(K_i;C_iN_0)+\Gamma_i(J_i,N_0)-\Gamma_i(J_i;N_0)\,L(J_i)=0$ by definition of K_i, N_0, hence $k_i\cdot n_0\in N_0$ as in (12). The final condition for (a) $K_i\cdot N\subset N$ is

$$(14)\qquad \Gamma_i(T_i(k_i\cdot n);\, C_i\cdot N_0)=0, \qquad \text{i.e., } T_i(k_i\cdot n)\in K_i.$$

For $m\in C_i\cdot N_0$, this follows from $\Gamma_i(T_i(k_i\cdot n);m)=L(T_i(T_i(k_i\cdot n)\cdot m))-L(T_i(m))\,L(T_i(k_i\cdot n)) = L(T_i(T_i(n)\cdot(k_i\cdot m)))-L(T_i(m))\,L(T_i(n))\,L(k_i)$ (because $\Gamma_i(k_i;n)=0$) $=L(T_i(k_i\cdot m))\,L(T_i(n))-L(T_i(m))\,L(T_i(n))\,L(k_i)$ (since $k_i\cdot m\in C_i\cdot N_0$ and so $\Gamma_i(T_i(n);k_i\cdot m)=0)=L(T_i(m))[L(k_i),\; L(T_i(n))]$ (because $\Gamma_i(k_i;m)=0$) $=0$ by (7), hence (14) holds and

(a) $K_i\cdot N\subset N$

follows from (10), (11), (14). Knowing (a), it is easy to establish

(b) $P(K_i)K_i\subset K_i$;

indeed for $x_i,\; y_i\in K_i$ and $m\in C_i\cdot N_0$ we have $L(T_i(P(x_i)\,y_i\cdot m))=$

$L(T_i(x_i\cdot(\bar{y}_i\cdot(x_i\cdot m))))=L(T_i(m))L(x_i)L(\bar{y}_i)L(x_i) = L(T_i(m))\ L(P(x_i)y_i)$. For (c) $P(N)K_i^\mu\subset K_i$ we first reduce the problem

$$(15)\qquad Q_i(N)\subset K_i\Rightarrow P(N)K_i^\mu\subset K_i.$$

Indeed, $L(T_i(P(n)k_i^\mu\cdot m)) = L(T_i(Q_i(n)\cdot(k_i\cdot m)))$ (by $n\in N$) $=L(T_i(k_i\cdot m))\,L(Q_i(n))$ (by assumption $Q_i(n)\in K_i$) $= L(T_i(m))\,L(k_i)\,L(Q_i(n))$ (by $k_i\in K_i$) $= L(T_i(m))\,L(Q_i(n))\,L(k_i)$ (by (7)) $= L(T_1(m))\,L(P(n)k_i^\mu)$ (by $n\in N$). To show $Q_i(n)\in K_i$ we need the general formula

$$(16)\qquad \Gamma_i(Q_i(m);m^*)\cdot u=-\Gamma_i(T_i(m);m)\cdot m^*\qquad (m\in M),$$

which holds because $\Gamma_i(Q_i(m);m^*)\cdot u=\Gamma_i(Q_i(m);m+m^*)\cdot u - \Gamma_i(Q_i(m);m)\cdot u = \Gamma_i(Q_i(m);T_i(m)\cdot u)\cdot u - T_i(Q_i(m)\cdot m)\cdot u + T_i(m)\cdot(Q_i(m)\cdot u)$ (by (1.6.3)) $=[L(Q_i(m)),L(T_i(m))]\cdot u-Q_i(m)\cdot m-(Q_i(m)\cdot m)^* + L(T_i(m))\,L(Q_i(m))u$ (by (8) and (1.6.3)) $=Q_i(m)\cdot(m+m^*)-Q_i(m)\cdot m - (Q_i(m)\cdot m)^*$ (by (1.6.3) and (1.3.10)) $= (Q_i(m)-Q_i(m^*))\cdot m^* = -\Gamma_i(T_i(m);m)m^*$ (by (3.9)). We also need the formula

$$(17)\qquad P(n)\,T_i(m)^\mu=T_i(Q_i(n)\cdot m)\qquad (n\in N,\ m\in C_i\cdot N_0).$$

By u-faithfulness it suffices to show that $[P(n)\,T_i(m)^\mu-T_i(Q_i(n)\cdot m)]\cdot u = Q_i(n)\cdot(m+m^*)-Q_i(n)\cdot m-(Q_i(n)\cdot m)^*$ (by $n\in N$, (1.6.3)) $=(Q_i(n)-Q_i(n)^*)\cdot m^*=\Gamma_i(Q_i(n);m^*)\cdot u$ (by (1.6.12)) vanishes. Now linearize $m\to n+\lambda m$ in (16) and take coefficients of λ to get $\Gamma_i(Q_i(n);m^*)u+\Gamma_i(Q_i(n,m);n^*)u = -\Gamma_i(T_i(n);n)\cdot m^*-\Gamma_i(T_i(n);m)n^*-\Gamma_i(T_i(m);n)n^*$. For $n\in N\subset N_0$ we have $\Gamma_i(J_i;n)=\Gamma_i(J_i;n^*)=\Gamma_i(T_i(n);m)=0$, so $\Gamma_i(Q_i(n);m^*)\cdot u=0$, establishing (17). We finally can prove

$$(18)\qquad Q_i(n)\subset K_i.$$

From (17) we have $\Gamma_i(Q_i(n);m)=L(T_i(Q_i(n)\cdot m))-L(T_i(m))\,L(Q_i(n)) = L(P(n)\,T_i(m)^\mu)-L(Q_i(n))\,L(T_i(m))+[L(Q_i(n)),L(T_i(m))] = \Delta_i(T_i(m);n) + \Gamma_i(Q_i(n);T_i(m)\cdot u)$ (by (8)) $=0+\Gamma_i(Q_i(n);m+m^*)$ (by $n\in N$, (1.6.3)), so $\Gamma_i(Q_i(n);m^*)=0$, and (18) follows because $(C_i\cdot N_0)^*=C_i\cdot N_0$. Note that by (15) and (18) we are finally done. ∎

3.11. *Remark.* Instead of symmetric definitions of N_0, N we can define them via $i=1$ alone:

(i) $\Gamma_1(J_1;m)=0\Rightarrow m\in N_0$,

(ii) $\Delta_1(J_1;m)=\Delta_1(J_1;m,N_0)=\Gamma_1(T_1(m);C_1\cdot N_0)=0\Rightarrow m\in N$.

Note $\Gamma_2(J_2; m)^* = \Gamma_1(J_1; m^*)$, and in general

$$\Gamma_i(x_i; m - m^*) = L(T_i(\Gamma_i(x_i; m) \cdot u))$$

by $L(T_i(x_i \cdot (m - m^*))) - L(T_i(m - m^*)) L(x_i) = L(T_i(x_i \cdot m - x_i^* \cdot m)) - 0$ (by (1.6.2)) $= L(T_i(\Gamma_i(x_i; m)u))$ (by (1.6.12)). In particular, $\Gamma_1(J_1; m) = 0$ implies $\Gamma_1(J_1; m^*) = 0$, establishing (i). For (ii) note that the last part of the assumption says $T_1(m) \in K_1$, hence $\Delta_2(J_2; m)^* = \Delta_1(J_1; m^*) = \Delta_1(J_1; T_1(m) \cdot u) - \Delta_1(J_1; m, T_1(m) \cdot u) + \Delta_1(J_1; m) = 0$ since $T_1(m) \in K_1 \Rightarrow T_1(m) \cdot u \in K_1 \cdot u \subset N$, also $\Delta_2(J_2; m, N_0)^* = \Delta_1(J_1; m^*, N_0^*) = \Delta_1(J_1;\ T_1(m) \cdot u, N_0) - \Delta_1(J_1; m, N_0) = 0$ using $N_0^* = N_0$, $T_1(m) \cdot u \in N$, and finally $\Gamma_2(T_2(m);\ C_2 \cdot N_0)^* = \Gamma_1(T_1(m^*);\ C_1 \cdot N_0) = 0$ because $T_1(m^*) = T_1(m)$. ∎

3.12. *Remark.* A slightly smaller Clifford subsystem is $J_{q0} = K_{10} + N_{00} + K_{20}$ for $K_{i0} = \{k_i; \Gamma_i(k_i; M) = 0\}$ and $N_{00} = \{n \in M; \Delta_i(J_i; n) = \Delta_i(J_i; n, N_0) = \Gamma_i(T_i(n); M) = 0\}$. ∎

3.13. Example. If $J = H_2(D, D_0, \pi, {}^-)$ is a hermitian matrix system as in Section 2, then

$$(*)\qquad \begin{aligned} \Delta_1(d_0[11]; d[12]) &= L_D(d[d_0, d^\pi]) \\ \Gamma_1(d_0[11]; d[12]) &= L_D([d_0, d]) \end{aligned}$$

for $L_D =$ left multiplication in D. With $Z = \{z \in D;\ [z, D_0] = 0\}$, the Clifford subsystem J_q of J constructed in (3.10) becomes

$$\begin{aligned} J_q &= H_2(Z, Z \cap D_0, \pi, {}^-) \\ &= (Z \cap D_0)[11] \oplus Z[12] \oplus (Z \cap D_0)[22]. \end{aligned}$$

(Note $N_0 = Z[12]$, whence $d_0[11] \in K_1$ iff $[d_0, \langle D_0\rangle Z] = 0$ for $\langle D_0\rangle$ the subalgebra of D generated by D_0; since $[d_0, \langle D_0\rangle Z] = \langle D_0\rangle[d_0, Z] + [d_0, \langle D_0\rangle]Z = [d_0, \langle D_0\rangle]Z$ we get $K_1 = (Z \cap D_0)[11]$. By (3.10.6) always $N \subset N_0 = Z[12]$, and $Z[12] \subset N$ easily follows from $(*)$.)

The Clifford subsystem J_{q0} of (3.12) will in general be smaller in this example: with $Z_0 = \{z \in D;\ [z, D] = 0\} \subset Z$ and $Y_0 = \{y \in D; [y, D_0] = 0 = [y + y^\pi, D]\} \subset Z$ we have

$$J_{q0} = (Z_0 \cap D_0)[11] \oplus Y_0[12] \oplus (Z_0 \cap D_0)[22].$$

Other Clifford subsystems can be obtained as follows: let E be a commutative subalgebra of $(D, \pi, {}^{-})$; then the subsystem $H_2(E, E\cap D_0, \pi, {}^{-})$ is Clifford by (3.6): $\Delta_1((E\cap D_0)[11]; E[12])\equiv 0$ by $(*)$. ∎

3.14. CLIFFORD SIMPLICITY CRITERION. *If C acts faithfully on M then an ample Clifford system*

$$J(q, S, C_0) = C_0 e_1 \oplus M \oplus C_0 e_2$$

is simple iff

(1) *q: $M \to C$ is nondegenerate and*

(2) *$(C, \ )$ is simple,*

in which case u is C-faithful.

Proof. We first show q is nondegenerate as soon as J has no proper ideal of trivial elements:

$(*)$ $\quad R = \operatorname{Rad} q = \{z\in M; q(z) = q(z, M) = 0\} \lhd J \quad$ has $\quad P(R)J = 0.$

Indeed, $CR\subset R$ by quadraticity of q $(q(cz) = c^2 q(z) = 0,\ q(cz, M) = cq(z, M) = 0)$ and $S(R) = R$ by q-orthogonality of S $(q(Sz) = \overline{q(z)} = 0,\ q(Sz, M) = q(Sz, SM) = \overline{q(z, M)} = 0)$, so $P(R)J = 0$ by (3.3.iv) (since $q(R) = q(R, S(n)) = 0$) and $P(J)R\subset R$ by (3.3.iv). Similarly, $\{JJR\}\subset R$ by (3.3.v). Since $R\subset M\subset J$, if J is simple we must have $R = 0$ as in (1).

To see that simplicity implies (2), suppose B were a proper ideal of $(C, {}^{-})$. Then $I = B_0 e_1 \oplus BM \oplus B_0 e_2$ is an ideal of J for $B_0 = B\cap C_0$: it is an ideal by (3.3) since the coefficients of e_i all have at least one factor from $B = \bar{B}$, and lie in C_0, hence lie in B_0 (note $S(BM) = \bar{B}M$, $q(M, BM) = Bq(M, M)$, $q(BM) = B^2 q(M)$), while all terms in M have at least one factor B $(q(BM) + q(BM, M)\subset B$ by the above). If J is simple then either $I = J$ (so $1\in B_0\subset B\lhd C \Rightarrow B = C$) or $I = 0$ (so $BM = 0 \Rightarrow B = 0$ by faithfulness of C on M).

Once $(C, {}^{-})$ is simple, the proper ideal $\operatorname{Ann}_C(u)\lhd (C, {}^{-})$ must vanish and u is C-faithful.

For the converse, suppose (1) and (2) hold. Any ideal $I\lhd J$ has Peirce decomposition $I = B_1 e_1 \oplus N \oplus B_2 e_2$. Now if $I\neq J = P(e)J$ then $e\notin I$, so $e_1\notin I$ (else $e = e_1 + P(u)e_1 \in I$), so no b_0 is invertible in C (if $b_0^{-1}\in C$ then $\bar{b}_0^{-2}\in (C)^2\cdot 1\subset C_0$ and $e_1 = b_0^2\overline{\bar{b}_0^{-2}}e_1 = P(b_0 e_1)(\bar{b}_0^{-2} e_1)\in P(I)J\subset I$). By (2) either $C = F$ or $(C, {}^{-}) = (F\boxplus F$, exchange) for a field F, so $\bar{B}_0 = B_0\subset C_0\subset C$ not invertible forces $B_0 = 0$ (easy if $C = F$, if $C = F\boxplus F$ then $b_0 = (\alpha, 0)$ or $(0, \alpha)$ so $B_0 \ni b_0 + \bar{b}_0 = (\alpha, \alpha)$ not invertible forces $\alpha = 0$); but then $q(N) + q(N, M)\subset B_0 = 0$ by (3.3) forces $N\subset \operatorname{Rad} q = 0$ by (1), so $B_0 = N = 0$ and $I = 0$. ∎

3.15. *Remark.* Since $(C, {}^{-})$ is simple iff either $C = F$ is a field with automorphism $^-$ or $C = F \boxplus \bar{F}$ is polarized for a field F and $^- =$ exchange automorphism, it follows that a simple $J(q, S, C_0)$ is either (I) $J(q, S, F_0)$ for a nondegenerate Q over a field F and an ample subspace $F_0 \subset F$, or (II) polarized $(J(q, S, F_0), J(q, S, F_0))$, where $J(q, S, F_0)$ is as in (I). ∎

Rather than assuming that the whole Clifford system $J(q, S, C_0)$ is simple (as we did in (3.14)), we can as well look at the case where only the Peirce-2-space C_0 is simple (which holds if J is simple by [6]). Note that a simple triple system always has characteristic 0 or p. If $1 \in q(M)$ (e.g., if J is triangulated), C_0 fulfills the assumption of

3.16. LEMMA (Rank-1-Simplicity). *If $C_0 \subset C$ is a $^-$-ample subspace $(1 \in C_0 = \bar{C}_0$, $cC_0c \subset C_0$ for all $c \in C)$ of a commutative associative algebra C with involution $^-$, then C_0 is a simple Jordan triple system under $P(c_0)\, d_0 = c_0\, \bar{d}_0 c_0$ iff $C' = C$ is $^-$-simple of characteristic $\neq 2$ or $C' = C/Z$ for $Z = \{z \in C; z^2 = 0\}$ is $^-$-simple of characteristic 2:*

(i) *$C' = F$ a field with involution $^-$ and $C_0 = F_0 = \bar{F}_0$ or*

(ii) *$C' = F \boxplus \bar{F}$ with exchange involution, $C_0 = F_0 \boxplus \bar{F}_0$ for an ample subspace $F_0 \subset F$.*

Proof. Suppose C_0 is simple and let M be a maximal ideal of $(C, {}^-)$. Since $M \cap C_0 \lhd C_0$ we have $M \cap C_0 = 0$ because otherwise $M \cap C_0 = C_0 \ni 1$. This shows $M = 0$ if $\frac{1}{2} \in k$ (then $C_0 = C$: $((c+1)^2 - c^2 - 1^2)1 = 2c$), and in characteristic 2 we have $m^2 = m1m \in M \cap C_0 = 0$, so $M \subset Z \subseteq C$, but certainly $Z \lhd (C, {}^-)$ (in characteristic 2!), so in either case C' is $^-$-simple, and $C' = F$ or $F \boxplus \bar{F}$ for a field F. In the latter case $1 = e_1 + \bar{e}_1$ forces $e_1 C_0 = e_1 C_0 e_1 \subset C_0$, so $C_0 = F_0 \boxplus \bar{F}_0$.

Conversely, if $(C', {}^-)$ is simple and $0 \neq I_0 \lhd C_0$ then I_0 contains an invertible element x (in case (ii) this follows from $\bar{I}_0 = P(1) I_0 \subset I_0$) and thus $c_0 = P(x)(x^{-1} c_0 x^{-1}) \in I_0$ for any $c_0 \in C_0$. ∎

4. SIMPLE TRIANGULATED TRIPLES

In this section we will classify simple triangulated Jordan triple systems by showing that such a triple is a hermitian matrix system or a Clifford system (it can be both). Our theorem generalizes the well-known Capacity Two Theorem for Jordan algebras which says that a simple Jordan algebra of capacity two is an outer ideal of a Jordan algebra $J(Q, 1)$ of a nondegenerate quadratic form with base point 1 or is an outer ideal of a hermitian matrix algebra $H_2(D, *)$. Although our result sounds very much like the Capacity Two Theorem, our assumptions are in fact much less restrictive: The former theorem assumes that the Peirce-2-spaces J_i, $i = 1, 2$, are division algebras, whereas we only know that the subsystems J_i are simple, and therefore may be any $H(A, \pi)$ for (A, π) a simple associative algebra. Nevertheless, we will use a trick which was repeatedly used in the usual proof of the Capacity Two Theorem:

4.1. *Isotope Trick.* Let $\mathscr{P}(J, m)$ be a property of elements $m \in M$ and $\mathscr{H}(J)$ a set of hypotheses on J defined in terms of the specialization of J_1 on M (without reference to u). Then if $\mathscr{H}(J) \Rightarrow \mathscr{P}(J, u)$ for all J, we also have $\mathscr{H}(J) \Rightarrow \mathscr{P}(J, m)$ for all invertible m and all J, since for such m the isotope $\tilde{J} = J^{(v)}$ $(v = e_1 + Q_2(m)^{-1})$ is triangulated by $\tilde{e}_1 = e_1$, $\tilde{e}_2 = Q_2(m)$, $\tilde{u} = m$ with the same $\tilde{J}_t = J_t$ and the same specialization of J_1 on M $(\tilde{L}(x_1) = L(x_1))$, so the hypotheses $\mathscr{H}$ continue to hold in $\tilde{J}$ and so imply $\mathscr{P}(\tilde{J}, \tilde{u})$, which means $\mathscr{P}(J, m)$ holds in J. ∎

Usually one seeks to establish such a property for *all m*, and this requires some sort of a "density" argument. If $\mathscr{P}$ were a polynomial and a suitable Zariski topology existed, the validity of $\mathscr{P}$ on the dense set of invertible elements would imply its validity everywhere. Thus in [3] one needed to show that in the capacity two case the invertible elements spanned M, except when $|J_1| \leqslant 3$, indeed if m, m' were invertible then there existed nonzero x_1, x_1' with $x_1 m_1 + x_1' m'$ invertible. A trick used by Zel'manov [8] provides a "density" principle even when u is the only invertible element of J: the idea is that u creates lots of invertible elements in the formal power series system.

4.2. *Laurent Trick.* Let $\mathscr{P}(J, m)$ be a property of elements $m \in M$ which is "linear": if $\mathscr{P}(\tilde{J}, \tilde{m})$ holds for $\tilde{m} = u + tm \in J[t] \subset \tilde{J}$ then $\mathscr{P}(J, m)$ holds. Let $\mathscr{H}(J)$ be a set of hypotheses on J which are inherited by $\tilde{J} = J[[t]]$ or $\tilde{J} = J((t))$. The element $\tilde{m}$ is invertible in $\tilde{J}$ (since $P(\tilde{m}) = P(u)(I + tP(u)\, P(u, m) + t^2 P(u)\, P(m))$ is invertible on $\tilde{J}$). Thus if $\mathscr{H}(J) \Rightarrow \mathscr{P}(J, m)$ for all invertible m and all J, then $\mathscr{H}(J) \Rightarrow \mathscr{P}(J, m)$ for all m and all J. ∎

The main work in classifying simple triangulated Jordan triples is done in

4.3. Proposition. *If J is a simple triangulated triple system, then C is $(\pi, {}^{-})$-simple with π-ample subspace C_0 and u is C-faithful. If $\pi \neq \mathrm{id}$ is nontrivial then $M = Cu$; if $\pi = \mathrm{id}$ then C is commutative.*

Proof. Everything will follow if we can show that $(C, \pi, {}^{-})$ is simple. Indeed, u automatically will be C-faithful: $Z = \{z \in C;\, zu = 0\}$ is a $(\pi, {}^{-})$-ideal since $\bar{z}u = \overline{zu} = 0$, $z^{\pi}u = -zu = 0$ by (1.6.8) and $zCu = zC^{*}u$ (by (1.6.10)) $= C^{*}zu = 0$, and $Z \neq C$ since $1 \notin Z$, so $Z = 0$. Moreover, if π is not trivial then the $(\pi, {}^{-})$-ideal

$$C' = C\{c - c^{\pi}\}C = C[C, C]C \tag{1}$$

is nonzero, hence all of C by $(\pi, {}^-)$-simplicity, and in general by (1.6.13):

(2) $$1 \in C' \Rightarrow M = Cu.$$

To show $(\pi, {}^-)$-simplicity of C let $R = R^\pi = \bar{R}$ be a maximal $(\pi, {}^-)$-ideal of C. In proving that $R = 0$ we will need to make temporary use of the fact that

(3) $$\tilde{C} = C/R \text{ is } (\pi, {}^-)-\text{simple}$$

(so the structure of $\tilde{C}$ is described in (2.8)). Our eventual goal $R = 0$ in $C \subset \mathrm{End}_k(M)$ means $RM = 0$; by nondegeneracy of Q_2 (by (1.15)), it will be enough to establish for all $r \in R$ and $m, n \in M$ that

(4) $$Q_2(rm) = 0,$$

(5) $$Q_2(rm, n) = 0.$$

From now on we will assume only that J_1 is simple (by (1.15)) and we will derive (4) and (5). Without much effort we can obtain these for $m = n = u$: The nonzero *-specialization L: $J_1 \to C_0$ is an isomorphism by simplicity of J_1, so $C_0 = L(J_1) \cong J_1$ is a simple Jordan triple system under $P(d_0)c_0 = d_0 \bar{c}_0 d_0$. Therefore the Jordan triple ideal $R \cap C_0$ vanishes. By (1.6.8) and (1.6.9), $r + r^\pi = L(T_1(ru))$, $rL(x_1)r^\pi = L(P(ru)x_1^\mu)$, and $rr^\pi = L(Q_1(ru))$ lie in $R \cap C_0 = 0$, so

(6) $$r^\pi = -r,\ r^2 = 0, \quad rL(x_1)r^\pi = 0 \qquad \text{for} \quad r \in R,\ x_1 \in J_1$$

(7) $$T_i(ru) = 0 = Q_i(ru) \qquad \text{for} \quad i = 1, 2$$

(we saw (7) for $i = 1$, and for $i = 2$ by skewness $(ru)^* = r^\pi u = -ru$ (by (1.6.10)) we see by (1.6.2) $0 = T_1(ru)^* = T_2((ru)^*) = -T_2(ru)$ and $0 = Q_1(ru)^* = Q_2((ru)^*) = Q_2(-ru) = Q_2(ru)$).

We want to establish the properties

$$(4_m)\ Q_2(rm) = 0, \quad (4\tfrac{1}{2}{}_m)\ Q_2(rm, m) = 0, \quad (5_m)\ Q_2(RM, m) = 0$$

for arbitrary $m \in M$. We avoid giving a direct proof, and instead reduce these to the case $m = u$ using our tricks (by (7) we already have established (4_u), $(4\tfrac{1}{2}{}_u)$). The hypothesis of Peirce-1-simplicity is inherited by $J((t))$ (since $J((t))_1 = J_1((t))$, here it is crucial to use Laurent series instead of just formal power series), so by the Laurent Trick if suffices to prove (4_m), $(4\tfrac{1}{2}{}_m)$, and (5_m) for all invertible m. These properties depend only on the representation of J_1 on M (this would not be true for $Q_1(m) = P(rm)e_2$

since this depends on e_2) and are independent of u, and so by the Isotope Trick it suffices to prove them for u. Thus (4_u), $(4\frac{1}{2}{}_u)$ guarantee that (4_m), $(4\frac{1}{2}{}_m)$ hold for all m and all that remains is to establish

$$(5_u) \qquad T_2(RM)=0.$$

At this point, for the second time we avoid a direct proof, and instead use the $(\pi, {}^-)$-simplicity of $\tilde{C}$. The $(\pi, {}^-)$-ideal $\tilde{C}'=\tilde{C}[\tilde{C},\tilde{C}]\tilde{C}$ of $\tilde{C}$ can only be $\tilde{C}$ or $\tilde{0}$. If $\tilde{C}'=\tilde{C}$ then $\tilde{1}\in\tilde{C}'$, and since R is nil this forces $1\in C'$ (if $1=c'+r$ for $c'\in C'$, $r\in R$ then $1=1^2=c'^2+c'r+rc'\in C'$ by (6)), hence $M=Cu$ by (2), in which case (5_u) reduces to $(4\frac{1}{2}{}_u)$: $T_2(RM)=T_2(RCu)\subset T_2(Ru)=Q_2(Ru,u)=0$ by $(4\frac{1}{2}{}_u)$. Thus in this case (4), (5) hold for all m.

If $\tilde{C}'=0$ then $[\tilde{C},\tilde{C}]=0$ and $\tilde{C}$ is commutative, so the reversal involution π is trivial, and $(\pi, {}^-)$-simplicity reduces to ${}^-$-simplicity. It is well known (and follows from (2.8)) that in this case either $\tilde{C}=F$ is a field with automorphism, or $C=F\boxplus\bar{F}$ is a direct sum of fields with exchange automorphism (involutory automorphism = involution for commutative algebras). In particular $\tilde{C}$ has no nilpotent elements, so neither does the subspace $\tilde{C}_0=C_0\cong J_1\cong J_2$. We already know $Q_2(rm)=0$ by (4), so $Q_1(rm)^2=P(rm)P(e_2)P(rm)e_1=P(rm)P(e_2)Q_2(rm)=0$ forces $Q_1(rm)=0$ too in the absence of nilpotents, therefore $T_2(rm)^2=P(\{rm,e_1,u\})e_2=[P(rm)P(e_1)P(u)+P(u)P(e_1)P(rm)+L(rm,e_1)P(u)L(e_1,rm)-P(P(rm)P(e_1)u,u)]e_2 = Q_2(rm)+Q_1(rm)^*+Q_2(rm,P(u)\overline{rm})-0$ (by $P(e_1)u=0$) $=0+0+Q_2(rm,(rm)^*)=Q_2(rm,r^*[T_1(m)\cdot u-m])$ (by (1.6.3)) $=Q_2(m,r^\pi L(T_1(m))r^*u)-Q_2(m,r^\pi r^*m) = -Q_2(m,rL(T_1(m))r^\pi u)+Q_2(m,rr^*m)$ (by $r^\pi=-r$ via (6), (1.6.10)) $=Q_2(m,rr^*m)$ (since $rL(T_1(m))r^\pi=0$ by (6)). If we can establish

$$(8) \qquad Q_2(m,rr^*m)=0 \qquad (r\in R,\ m\in M)$$

then $T_2(rm)^2=0$ will force $T_2(rm)=0$ in J_2, and we will have established (5_u), completing the verification of (4), (5). (Note that (8) does not follow from $(4\frac{1}{2}{}_m)$ since $rr^*\notin R$ because $r^*\in C_2$ doesn't lie in C.) Now we can apply the Laurent Trick (4.2) to reduce the property

$$(8_m) \qquad Q_2(m,RC^*m)=0$$

from arbitrary m to invertible m. But for invertible m we have $C^*m\subset Cm$ using $L(x_2)m=P(x_2,m)e_2 = P(P(m)P(m)^{-1}x_2,m)e_2=L(m,P(m)^{-1}x_2)P(m)e_2 = \{Q_1(m),P(m)^{-1}x_2,m\}=L(Q_1(m))L(P(m)^{-1}x_2)m\in Cm$, so $Q_2(m,RC^*m)\subset Q_2(m,RCm)\subset Q_2(m,Rm)=0$ by $(4\frac{1}{2}{}_m)$ (which is why we brought this formula along!). ∎

Remark. We could give a more direct proof of (4), (5) (i.e., of the passage from $m=u$ to general m) if we could establish

$$\text{(A?)} \qquad Q_1(cm) = P_{cu} Q_1(m)^\mu$$

$$\text{(B?)} \qquad Q_2(cm) = P_m Q_2(cu)^\mu.$$

Indeed, then $Q_2(rm) = P_m Q_2(ru)^\mu = 0$ by (7), and $Q_1(rm)$ vanishes since $Q_1(rm)^2 = [P(ru)\, P(Q_1(m)^\mu)\, P(ru)]\, e_1 = P(ru)\, P(Q_1(m)^\mu)\, Q_2(ru) = 0$. Here (A), (B) are easy to establish for monomials, and the mixed terms c, d follow from the special cases $d=1$:

$$\text{(A}'\text{)} \qquad Q_1(cm, m) = P(cu, u)\, Q_1(m)^\mu$$

$$\text{(B}'\text{)} \qquad Q_2(cm, m) = P_m T_2(cu)^\mu$$

(for the first we induct on the length of d, using (1.3.8)). We can establish (B′) for monomials of odd length in characteristic $\neq 2$, since if $D(c, m)$ is the difference of the two terms we have $D(x_1 c_0 + c_0 x_1, m) = 0$ so $D(x_1 \cdots x_r, m)$ changes sign under cyclic permutation. However, we have not been able to establish (A′) or (B′) in general. ∎

4.4. THEOREM. *A triangulated Jordan triple system is simple iff it is isomorphic to one of the following*:

(I) $H_2(D, D_0, \pi, {}^-)$ *for simple noncommutative* D *with* π^-;

(II) $M_2(D, {}^-)$ *for simple noncommutative* D *with involutory automorphism* ${}^-$, $P(x)\, y = x\bar{y}x$ *for* $\overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}$;

(III) $M_2(D, \iota)$ *for simple noncommutative* D *with involution* ι, $P(x)\, y = x\bar{y}'x$ *for* $\overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \begin{pmatrix} \iota(a) & \iota(b) \\ \iota(c) & \iota(d) \end{pmatrix}$;

(IV) *polarized* $(H_2(B, B_0, \pi), H_2(B, B_0, \pi))$ *for a simple noncommutative* B *with involution*;

(V) *polarized* $(M_2(A), M_2(A))$ *for a simple noncommutative* A;

(VI) $J(q, S, F_0)$ *for a nondegenerate* Q *over a field* F *and* $F_0 \subset F$ *ample*;

(VII) *polarized* $(J(q, S, F_0), J(q, S, F_0))$ *for* $J(q, S, F_0)$ *as in* (VI).

Proof. By (4.3) we know C is $(\pi, {}^-)$-simple, u is C-faithful, and either $\pi = 1$ (C is commutative) or $\pi \neq 1$ (C is not commutative, $M = Cu$). In the latter case $J = J_h \cong H_2(C, C_0, \pi, {}^-)$ ($C = D$ here) by (2.4) and J has the form (I)–(V) in view of (2.10). In the former case (C commutative, $\pi = 1$), the cases (II), (III), and (V) of (2.8) disappear, so either (I) $C = F$ is a field with ${}^-$ or (IV) $C = F \boxplus \bar{F}$ with exchange automorphism. It suffices to prove $J = J_q$ (then (I), (IV) give cases (VI), (VII) above), and by the Clifford criterion (3.6) and the ($\Gamma \Rightarrow \Delta$) lemma 3.8 it suffices by C-faithfulness to verify

$$(x_1 - x_1^*) \cdot m = 0 \tag{1}$$

for all $x_1 \in J_1$, $m \in M$. Now by (1.6.12), $(x_1 - x_1^*) \cdot m = \Gamma_1(x_1; m)u$ for $\Gamma_1(x_1; m) = L(T_1(x_1 \cdot m)) - L(T_1(m))\, L(x_1) \in C$. Since C is commutative, (1.6.14) shows $\Gamma_1(x_1; m)^2 m = 0$, so $\Gamma_1(x_1; m)$ is never invertible. This immediately settles case (I), where $C = F$ is a field, and also case (IV) $C = F \boxplus \bar{F}$ if we want to invoke (1.16): J is polarized, so $\Gamma_1(x_1; m) = \Gamma_1(x_1^+; m^+) \boxplus \Gamma_1(x_1^-; m^-)$ vanishes because we have $\Gamma_1(x_1^\sigma; m^\sigma)^2 m^\sigma = 0$ for each summand. However, we prefer a more direct argument: If the non-invertible element $\Gamma_1(x_1; m)$ is invariant under $\bar{\ }$, it is zero, so (1) is clear for symmetric elements $x_1 = \bar{x}_1$, $m = \bar{m}$. But this already implies (1) in general: We have $1 = \varepsilon + \bar{\varepsilon}$ for orthogonal idempotents ε and $\bar{\varepsilon}$ in C, so $m = 1m = \varepsilon m + \bar{\varepsilon} m = \varepsilon(\varepsilon m + \overline{\varepsilon m}) + \bar{\varepsilon}(\bar{\varepsilon} m + \varepsilon \bar{m}) = \varepsilon h_1 + \bar{\varepsilon} h_2$ for $h_i = \bar{h}_i \in M$ and $L(x_1) = \varepsilon(\varepsilon L(x_1) + \bar{\varepsilon} L(\bar{x}_1)) + \bar{\varepsilon}(\bar{\varepsilon} L(x_1) + \varepsilon L(\bar{x}_1)) = \varepsilon L(y_1) + \bar{\varepsilon} L(z_1)$ for $y_1 = \bar{y}_1$ and $z_1 = \bar{z}_1 \in J_1$ since $\varepsilon L(x_1) = \varepsilon L(x_1) \varepsilon^\pi \in C_0$ by ampleness. Also $\varepsilon^* u = \varepsilon^\pi u = \varepsilon u$ forces $\varepsilon = \varepsilon^*$ by C-faithfulness, therefore $(x_1 - x_1^*) \cdot m = \{\varepsilon(y_1 - y_1^*) + \bar{\varepsilon}(z_1 - z_1^*)\} \cdot (\varepsilon h_1 + \bar{\varepsilon} h_2) = \varepsilon(y_1 - y_1^*) \cdot h_1 + \bar{\varepsilon}(z_1 - z_1^*) \cdot h_2 = 0$ by the symmetric case.

References

1. H. Hanche-Olsen and E. Størmer, "Jordan Operator Algebras," Monographs and Studies in Mathematics 21, Pitman, New York/London, 1984.
2. N. Jacobson, "Structure and Representations of Jordan Algebras," Amer. Math. Soc. Colloquium Publications, Vol. 39, Amer. Math. Soc., Providence, RI, 1968.
3. N. Jacobson, "Lectures on Quadratic Jordan Algebras," Tata Institute, Bombay, 1969.
4. N. Jacobson, "Structure Theory of Jordan Algebras," The University of Arkansas Lecture Notes, Vol. 5, Fayetteville, 1981.
5. O. Loos, "Jordan Pairs," Lecture Notes in Mathematics, Vol. 460, Springer-Verlag, New York/Berlin, 1975.
6. K. McCrimmon, Peirce ideals in Jordan triple systems, *Pacific J. Math.* **83** (1979), 415–439.
7. K. McCrimmon and K. Meyberg, Coordinatization of Jordan triple systems, *Comm. Algebra* **9** (1981), 1495–1542.
8. K. McCrimmon and E. Zel'manov, The structure of prime quadratic Jordan algebras, *Advances in Math.*, to appear.
9. K. Meyberg, Lectures on algebras and triple systems, Lecture Notes, University of Virginia, Charlottesville, 1972.
10. E. Neher, "Jordan Triple Systems by the Grid Approach," Lecture Notes in Mathematics, Springer-Verlag, New York/Berlin, 1987.